Problem Solutions for Chapter 2

2-1.
$$\mathbf{E} = 100\cos\left(2\pi 10^8 t + 30°\right)\mathbf{e}_x + 20\cos\left(2\pi 10^8 t - 50°\right)\mathbf{e}_y$$
$$+\ 40\cos\left(2\pi 10^8 t + 210°\right)\mathbf{e}_z$$

2-2. The general form is:

y = (amplitude) $\cos(\omega t - kz) = A\cos[2\pi(\nu t - z/\lambda)]$. Therefore

(a) amplitude = 8 μm

(b) wavelength: $1/\lambda = 0.8\ \mu m^{-1}$ so that $\lambda = 1.25\ \mu m$

(c) $\omega = 2\pi\nu = 2\pi(2) = 4\pi$

(d) At $t = 0$ and $z = 4\ \mu m$ we have

$$y = 8\cos[2\pi(-0.8\ \mu m^{-1})(4\ \mu m)]$$
$$= 8\cos[2\pi(-3.2)] = 2.472$$

2-3. For E in electron volts and λ in μm we have $E = \dfrac{1.240}{\lambda}$

(a) At 0.82 μm, E = 1.240/0.82 = 1.512 eV

At 1.32 μm, E = 1.240/1.32 = 0.939 eV

At 1.55 μm, E = 1.240/1.55 = 0.800 eV

(b) At 0.82 μm, $k = 2\pi/\lambda = 7.662\ \mu m^{-1}$

At 1.32 μm, $k = 2\pi/\lambda = 4.760\ \mu m^{-1}$

At 1.55 μm, $k = 2\pi/\lambda = 4.054\ \mu m^{-1}$

2-4. $x_1 = a_1\cos(\omega t - \delta_1)$ and $x_2 = a_2\cos(\omega t - \delta_2)$

Adding x_1 and x_2 yields

$$x_1 + x_2 = a_1[\cos\omega t\ \cos\delta_1 + \sin\omega t\ \sin\delta_1]$$
$$+ a_2[\cos\omega t\ \cos\delta_2 + \sin\omega t\ \sin\delta_2]$$
$$= [a_1\cos\delta_1 + a_2\cos\delta_2]\cos\omega t + [a_1\sin\delta_1 + a_2\sin\delta_2]\sin\omega t$$

Since the a's and the δ's are constants, we can set

$$a_1\cos\delta_1 + a_2\cos\delta_2 = A\cos\phi \qquad (1)$$

$$a_1 \sin \delta_1 + a_2 \sin \delta_2 = A \sin \phi \qquad (2)$$

provided that constant values of A and ϕ exist which satisfy these equations. To verify this, first square both sides and add:

$$A^2 (\sin^2 \phi + \cos^2 \phi) = a_1^2\left(\sin^2 \delta_1 + \cos^2 \delta_1\right)$$

$$+ a_2^2\left(\sin^2 \delta_2 + \cos^2 \delta_2\right) + 2a_1a_2 (\sin \delta_1 \sin \delta_2 + \cos \delta_1 \cos \delta_2)$$

or

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos (\delta_1 - \delta_2)$$

Dividing (2) by (1) gives

$$\tan \phi = \frac{a_1 \sin \delta_1 + a_2 \sin \delta_2}{a_1 \cos \delta_1 + a_2 \cos \delta_2}$$

Thus we can write

$$x = x_1 + x_2 = A \cos \phi \cos \omega t + A \sin \phi \sin \omega t = A \cos(\omega t - \phi)$$

2-5. First expand Eq. (2-3) as

$$\frac{E_y}{E_{0y}} = \cos (\omega t - kz) \cos \delta - \sin (\omega t - kz) \sin \delta \qquad (2.5\text{-}1)$$

Subtract from this the expression

$$\frac{E_x}{E_{0x}} \cos \delta = \cos (\omega t - kz) \cos \delta$$

to yield

$$\frac{E_y}{E_{0y}} - \frac{E_x}{E_{0x}} \cos \delta = - \sin (\omega t - kz) \sin \delta \qquad (2.5\text{-}2)$$

Using the relation $\cos^2 \alpha + \sin^2 \alpha = 1$, we use Eq. (2-2) to write

$$\sin^2(\omega t - kz) = [1 - \cos^2(\omega t - kz)] = \left[1 - \left(\frac{E_x}{E_{0x}}\right)^2\right] \qquad (2.5\text{-}3)$$

Squaring both sides of Eq. (2.5-2) and substituting it into Eq. (2.5-3) yields

$$\left[\frac{E_y}{E_{0y}} - \frac{E_x}{E_{0x}}\cos\delta\right]^2 = \left[1 - \left(\frac{E_x}{E_{0x}}\right)^2\right]\sin^2\delta$$

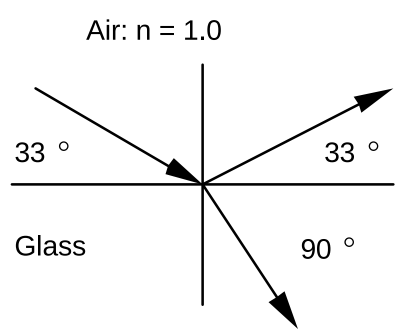
Expanding the left-hand side and rearranging terms yields

$$\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 - 2\left(\frac{E_x}{E_{0x}}\right)\left(\frac{E_y}{E_{0y}}\right)\cos\delta = \sin^2\delta$$

2-6. Plot of Eq. (2-7).

2-7. Linearly polarized wave.

2-8.



(a) Apply Snell's law

$$n_1 \cos\theta_1 = n_2 \cos\theta_2$$

where $n_1 = 1$, $\theta_1 = 33°$, and $\theta_2 = 90° - 33° = 57°$

$$\therefore\ n_2 = \frac{\cos\ 33°}{\cos\ 57°} = 1.540$$

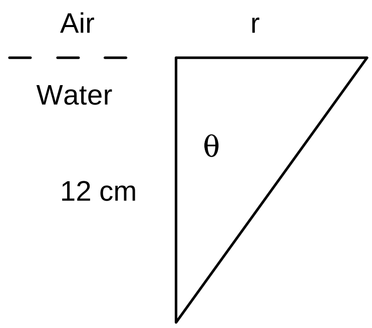
(b) The critical angle is found from

$$n_{glass} \sin\phi_{glass} = n_{air} \sin\phi_{air}$$

with $\phi_{air} = 90°$ and $n_{air} = 1.0$

$$\therefore \phi_{critical} = \arcsin \frac{1}{n_{glass}} = \arcsin \frac{1}{1.540} = 40.5°$$

2-9

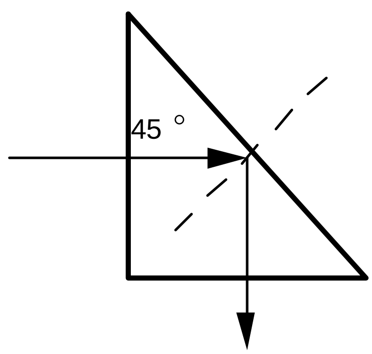


Find θ_c from Snell's law $\quad n_1 \sin\theta_1 = n_2 \sin\theta_c = 1$

When $n_2 = 1.33$, then $\theta_c = 48.75°$

Find r from $\tan\theta_c = \dfrac{r}{12\ \text{cm}}$, which yields r = 13.7 cm.

2-10.



Using Snell's law $\quad n_{glass} \sin\theta_c = n_{alcohol} \sin 90°$

where $\theta_c = 45°$ we have

$$n_{glass} = \frac{1.45}{\sin 45°} = 2.05$$

2-11. (a) Use either $NA = \left(n_1^2 - n_2^2\right)^{1/2} = \underline{0.242}$

or

$$NA \approx n_1 \sqrt{2\Delta} = n_1 \sqrt{\frac{2(n_1 - n_2)}{n_1}} = \underline{0.243}$$

(b) $\theta_{0,max} = \arcsin(NA/n) = \arcsin\left(\frac{0.242}{1.0}\right) = 14°$

2-13. $NA = \left(n_1^2 - n_2^2\right)^{1/2} = \left[n_1^2 - n_1^2(1-\Delta)^2\right]^{1/2}$

$$= n_1 \left(2\Delta - \Delta^2\right)^{1/2}$$

Since $\Delta << 1$, $\Delta^2 << \Delta$; $\therefore$ $NA \approx n_1 \sqrt{2\Delta}$

2-14. (a) Solve Eq. (2-34a) for jH_ϕ:

$jH_\phi = j\,\frac{\varepsilon\omega}{\beta}\,E_r - \frac{1}{\beta r}\frac{\partial H_z}{\partial \phi}$ Substituting into Eq. (2-33b) we have

$$j\,\beta\,E_r + \frac{\partial E_z}{\partial r} = \omega\mu\left[j\frac{\varepsilon\omega}{\beta}E_r - \frac{1}{\beta r}\frac{\partial H_z}{\partial \phi}\right]$$

Solve for E_r and let $q^2 = \omega^2\varepsilon\mu - \beta^2$ to obtain Eq. (2-35a).

(b) Solve Eq. (2-34b) for jH_r:

$jH_r = -j\,\frac{\varepsilon\omega}{\beta}\,E_\phi - \frac{1}{\beta}\frac{\partial H_z}{\partial r}$ Substituting into Eq. (2-33a) we have

$$j\,\beta\,E_\phi + \frac{1}{r}\frac{\partial E_z}{\partial \phi} = -\omega\mu\left[-j\frac{\varepsilon\omega}{\beta}E_\phi - \frac{1}{\beta}\frac{\partial H_z}{\partial r}\right]$$

Solve for E_ϕ and let $q^2 = \omega^2\varepsilon\mu - \beta^2$ to obtain Eq. (2-35b).

(c) Solve Eq. (2-34a) for jE_r:

$$jE_r = \frac{1}{\varepsilon\omega}\frac{1}{r}\left(\frac{\partial H_z}{\partial \phi} + jr\beta H_\phi\right)$$ Substituting into Eq. (2-33b) we have

$$\frac{\beta}{\varepsilon\omega}\frac{1}{r}\left(\frac{\partial H_z}{\partial \phi} + jr\beta H_\phi\right) + \frac{\partial E_z}{\partial r} = j\omega\mu\, H_\phi$$

Solve for H_ϕ and let $q^2 = \omega^2\varepsilon\mu - \beta^2$ to obtain Eq. (2-35d).

(d) Solve Eq. (2-34b) for jE_ϕ

$$jE_\phi = -\frac{1}{\varepsilon\omega}\left(j\beta H_r + \frac{\partial H_z}{\partial r}\right)$$ Substituting into Eq. (2-33a) we have

$$\frac{1}{r}\frac{\partial E_z}{\partial \phi} - \frac{\beta}{\varepsilon\omega}\left(j\beta H_r + \frac{\partial H_z}{\partial r}\right) = -j\omega\mu\, H_r$$

Solve for H_r to obtain Eq. (2-35c).

(e) Substitute Eqs. (2-35c) and (2-35d) into Eq. (2-34c)

$$-\frac{j}{q^2}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(\beta\frac{\partial H_z}{\partial \phi} + \varepsilon\omega r\frac{\partial E_z}{\partial r}\right) - \frac{\partial}{\partial \phi}\left(\beta\frac{\partial H_z}{\partial r} - \frac{\varepsilon\omega}{r}\frac{\partial E_z}{\partial \phi}\right)\right] = j\varepsilon\omega E_z$$

Upon differentiating and multiplying by $jq^2/\varepsilon\omega$ we obtain Eq. (2-36).

(f) Substitute Eqs. (2-35a) and (2-35b) into Eq. (2-33c)

$$-\frac{j}{q^2}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(\beta\frac{\partial E_z}{\partial \phi} - \mu\omega r\frac{\partial H_z}{\partial r}\right) - \frac{\partial}{\partial \phi}\left(\beta\frac{\partial E_z}{\partial r} + \frac{\mu\omega}{r}\frac{\partial H_z}{\partial \phi}\right)\right] = -j\mu\omega H_z$$

Upon differentiating and multiplying by $jq^2/\varepsilon\omega$ we obtain Eq. (2-37).

2-15. For $\nu = 0$, from Eqs. (2-42) and (2-43) we have

$E_z = AJ_0(ur)\, e^{j(\omega t-\beta z)}$ and $H_z = BJ_0(ur)\, e^{j(\omega t-\beta z)}$

We want to find the coefficients A and B. From Eqs. (2-47) and (2-51), respectively, we have

$$C = \frac{J_\nu(ua)}{K_\nu(wa)}\, A \quad \text{and} \quad D = \frac{J_\nu(ua)}{K_\nu(wa)}\, B$$

Substitute these into Eq. (2-50) to find B in terms of A:

$$A\left(\frac{j\beta\nu}{a}\right)\left(\frac{1}{u^2}+\frac{1}{w^2}\right) = B\omega\mu\left[\frac{J'_\nu(ua)}{uJ_\nu(ua)}+\frac{K'_\nu(wa)}{wK_\nu(wa)}\right]$$

For $\nu = 0$, the right-hand side must be zero. Also for $\nu = 0$, either Eq. (2-55a) or (2-56a) holds. Suppose Eq. (2-56a) holds, so that the term in square brackets on the right-hand side in the above equation is not zero. Then we must have that B = 0, which from Eq. (2-43) means that $H_z = 0$. Thus Eq. (2-56) corresponds to TM_{0m} modes.

For the other case, substitute Eqs. (2-47) and (2-51) into Eq. (2-52):

$$0 = \frac{1}{u^2}\left[B\frac{j\beta\nu}{a}J_\nu(ua) + A\omega\varepsilon_1 uJ'_\nu(ua)\right]$$

$$+\ \frac{1}{w^2}\left[B\frac{j\beta\nu}{a}J_\nu(ua) + A\omega\varepsilon_2 w\frac{K'_\nu(wa)J_\nu(ua)}{K_\nu(wa)}\right]$$

With $k_1^2 = \omega^2\mu\varepsilon_1$ and $k_2^2 = \omega^2\mu\varepsilon_2$ rewrite this as

$$B\nu = \frac{ja}{\beta\omega\mu}\left[\frac{1}{\dfrac{1}{u^2}+\dfrac{1}{w^2}}\right]\left[k_1^2 J_\nu + k_2^2 K_\nu\right] A$$

where J_ν and K_ν are defined in Eq. (2-54). If for $\nu = 0$ the term in square brackets on the right-hand side is non-zero, that is, if Eq. (2-56a) does not hold, then we must have that A = 0, which from Eq. (2-42) means that $E_z = 0$. Thus Eq. (2-55) corresponds to TE_{0m} modes.

2-16. From Eq. (2-23) we have

$$\Delta = \frac{n_1^2 - n_2^2}{2n_1^2} = \frac{1}{2}\left(1 - \frac{n_2^2}{n_1^2}\right)$$

$$\Delta << 1 \quad \text{implies } n_1 \approx n_2$$

Thus using Eq. (2-46), which states that $n_2k = k_2 \le \beta \le k_1 = n_1k$, we have

$$n_2^2k^2 = k_2^2 \approx n_1^2k^2 = k_1^2 \approx \beta^2$$

2-17.

2-18. (a) From Eqs. (2-59) and (2-61) we have

$$M \approx \frac{2\pi^2a^2}{\lambda^2}\left(n_1^2 - n_2^2\right) = \frac{2\pi^2a^2}{\lambda^2}(NA)^2$$

$$a = \left(\frac{M}{2\pi}\right)^{1/2}\frac{\lambda}{NA} = \left(\frac{1000}{2}\right)^{1/2}\frac{0.85\mu m}{0.2\pi} = 30.25\mu m$$

Therefore, D = 2a =60.5 μm

(b) $M = \dfrac{2\pi^2(30.25\mu m)^2}{(1.32\mu m)^2}(0.2)^2 = 414$

(c) At 1550 nm, M = 300

2-19. From Eq. (2-58),

$$V = \frac{2\pi\ (25\ \mu m)}{0.82\ \mu m}\ \left[(1.48)^2 - (1.46)^2\right]^{1/2} = 46.5$$

Using Eq. (2-61) $M \approx V^2/2$ =1081 at 820 nm.

Similarly, M = 417 at 1320 nm and M = 303 at 1550 nm. From Eq. (2-72)

$$\left(\frac{P_{clad}}{P}\right)_{total} \approx \frac{4}{3} M^{-1/2} = \frac{4 \times 100\%}{3\sqrt{1080}} = 4.1\%$$

at 820 nm. Similarly, $(P_{clad}/P)_{total}$ = 6.6% at 1320 nm and 7.8% at 1550 nm.

2-20 (a) At 1320 nm we have from Eqs. (2-23) and (2-57) that V = 25 and M = 312.

(b) From Eq. (2-72) the power flow in the cladding is 7.5%.

2-21. (a) For single-mode operation, we need $V \le 2.40$.

Solving Eq. (2-58) for the core radius a

$$a = \frac{V\lambda}{2\pi}\left(n_1^2 - n_2^2\right)^{-1/2} = \frac{2.40(1.32\mu m)}{2\pi\left[(1.480)^2 - (1.478)^2\right]^{1/2}} = 6.55\ \mu m$$

(b) From Eq. (2-23)

$$NA = \left(n_1^2 - n_2^2\right)^{1/2} = \left[(1.480)^2 - (1.478)^2\right]^{1/2} = 0.077$$

(c) From Eq. (2-23), NA = n sin $\theta_{0,max}$. When n = 1.0 then

$$\theta_{0,max} = \arcsin\left(\frac{NA}{n}\right) = \arcsin\left(\frac{0.077}{1.0}\right) = 4.4°$$

2-22. $n_2 = \sqrt{n_1^2 - NA^2} = \sqrt{(1.458)^2 - (0.3)^2} = 1.427$

$$a = \frac{\lambda V}{2\pi NA} = \frac{(1.30)(75)}{2\pi(0.3)} = 52\ \mu m$$

2-23. For small values of Δ we can write $V \approx \frac{2\pi a}{\lambda} n_1 \sqrt{2\Delta}$

For a = 5 μm we have $\Delta \approx 0.002$, so that at 0.82 μm

$$V \approx \frac{2\pi\ (5\ \mu m)}{0.82\ \mu m} 1.45 \sqrt{2(0.002)} = 3.514$$

Thus the fiber is no longer single-mode. From Figs. 2-18 and 2-19 we see that the LP_{01} and the LP_{11} modes exist in the fiber at 0.82 μm.

2-24.

2-25. From Eq. (2-77) $L_p = \frac{2\pi}{\beta} = \frac{\lambda}{n_y - n_x}$

For L_p = 10 cm $\quad n_y - n_x = \frac{1.3\times10^{-6}\ m}{10^{-1}\ m} = 1.3\times10^{-5}$

For L_p = 2 m $\quad n_y - n_x = \frac{1.3\times10^{-6}\ m}{2\ m} = 6.5\times10^{-7}$

Thus

$$6.5\times10^{-7} \le n_y - n_x \le 1.3\times10^{-5}$$

2-26. We want to plot n(r) from n_2 to n_1. From Eq. (2-78)

$$n(r) = n_1 \left[1 - 2\Delta(r/a)^{\alpha}\right]^{1/2} = 1.48 \left[1 - 0.02(r/25)^{\alpha}\right]^{1/2}$$

n_2 is found from Eq. (2-79): $\quad n_2 = n_1(1 - \Delta) = 1.465$

2-27. From Eq. (2-81)

$$M = \frac{\alpha}{\alpha+2}\ a^2k^2n_1^2\Delta = \frac{\alpha}{\alpha+2}\left(\frac{2\pi a n_1}{\lambda}\right)^2 \Delta$$

where

$$\Delta = \frac{n_1 - n_2}{n_1} = 0.0135$$

At λ = 820 nm, M = 543 and at λ = 1300 nm, M = 216.

For a step index fiber we can use Eq. (2-61)

$$M_{step} \approx \frac{V^2}{2} = \frac{1}{2}\left(\frac{2\pi a}{\lambda}\right)^2 \left(n_1^2 - n_2^2\right)$$

At λ = 820 nm, M_{step} = 1078 and at λ = 1300 nm, M_{step} = 429.

Alternatively, we can let $\alpha = \infty$ in Eq. (2-81):

$$M_{step} = \left(\frac{2\pi a n_1}{\lambda}\right)^2 \Delta = \begin{cases} 1086 \text{ at } 820 \text{ nm} \\ 432 \text{ at } 1300 \text{ nm} \end{cases}$$

2-28. Using Eq. (2-23) we have

$$\text{(a) NA} = \left(n_1^2 - n_2^2\right)^{1/2} = \left[(1.60)^2 - (1.49)^2\right]^{1/2} = 0.58$$

$$\text{(b) NA} = \left[(1.458)^2 - (1.405)^2\right]^{1/2} = 0.39$$

2-29. (a) From the Principle of the Conservation of Mass, the volume of a preform rod section of length $L_{preform}$ and cross-sectional area A must equal the volume of the fiber drawn from this section. The preform section of length $L_{preform}$ is drawn into a fiber of length L_{fiber} in a time t. If S is the preform feed speed, then $L_{preform}$ = St. Similarly, if s is the fiber drawing speed, then L_{fiber} = st. Thus, if D and d are the preform and fiber diameters, respectively, then

$$\text{Preform volume} = L_{preform}(D/2)^2 = St\,(D/2)^2$$

and

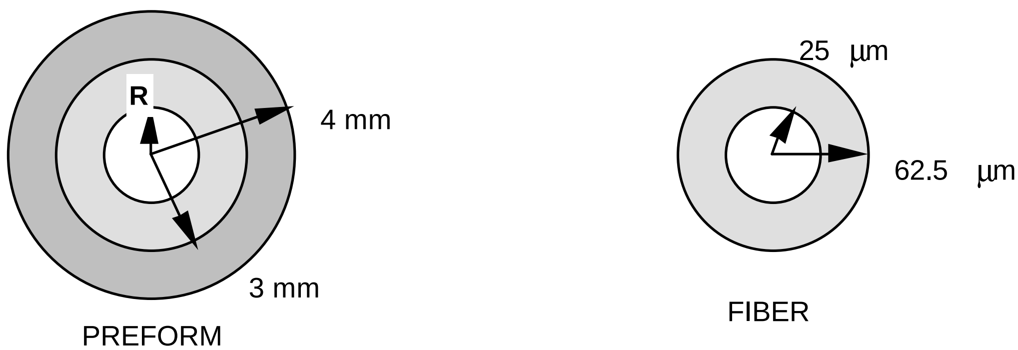
$$\text{Fiber volume} = L_{fiber}\,(d/2)^2 = st\,(d/2)^2$$

Equating these yields

$$St\left(\frac{D}{2}\right)^2 = st\left(\frac{d}{2}\right)^2 \quad \text{or} \quad s = S\left(\frac{D}{d}\right)^2$$

(b) $$S = s\left(\frac{d}{D}\right)^2 = 1.2 \text{ m/s} \left(\frac{0.125 \text{ mm}}{9 \text{ mm}}\right)^2 = 1.39 \text{ cm/min}$$

2-30. Consider the following geometries of the preform and its corresponding fiber:



We want to find the thickness of the deposited layer (3 mm - R). This can be done by comparing the ratios of the preform core-to-cladding cross-sectional areas and the fiber core-to-cladding cross-sectional areas:

$$\frac{A_{\text{preform core}}}{A_{\text{preform clad}}} = \frac{A_{\text{fiber core}}}{A_{\text{fiber clad}}}$$

or

$$\frac{\pi(3^2 - R^2)}{\pi(4^2 - 3^2)} = \frac{\pi\,(25)^2}{\pi\,\left[(62.5)^2 - (25)^2\right]}$$

from which we have

$$R = \left[9 - \frac{7(25)^2}{(62.5)^2 - (25)^2}\right]^{1/2} = 2.77 \text{ mm}$$

Thus, thickness = 3 mm - 2.77 mm = 0.23 mm.

2-31. (a) The volume of a 1-km-long 50-µm diameter fiber core is

$$V = \pi r^2 L = \pi\,(2.5\times10^{-3} \text{ cm})^2\,(10^5 \text{ cm}) = 1.96 \text{ cm}^3$$

The mass M equals the density ρ times the volume V:

$$M = \rho V = (2.6 \text{ gm/cm}^3)(1.96 \text{ cm}^3) = 5.1 \text{ gm}$$

(b) If R is the deposition rate, then the deposition time t is

$$t = \frac{M}{R} = \frac{5.1\ \text{gm}}{0.5\ \text{gm/min}} = 10.2\ \text{min}$$

2-32. Solving Eq. (2-82) for χ yields

$$\chi = \left(\frac{K}{Y\sigma}\right)^2 \qquad \text{where } Y = \sqrt{\pi} \qquad \text{for surface flaws.}$$

Thus

$$\chi = \frac{(20\ \text{N/mm}^{3/2})^2}{(70\ \text{MN/m}^2)^2\ \pi} = 2.60\times10^{-4}\ \text{mm} = 0.26\ \mu\text{m}$$

2-33. (a) To find the time to failure, we substitute Eq. (2-82) into Eq. (2-86) and integrate (assuming that σ is independent of time):

$$\int_{\chi_i}^{\chi_f} \chi^{-b/2}\, d\chi = AY^b\sigma^b \int_0^t dt$$

which yields

$$\frac{1}{1-\frac{b}{2}}\left[\chi_f^{1-b/2} - \chi_i^{1-b/2}\right] = AY^b\sigma^b t$$

or

$$t = \frac{2}{(b-2)A(Y\sigma)^b}\left[\chi_i^{(2-b)/2} - \chi_f^{(2-b)/2}\right]$$

(b) Rewriting the above expression in terms of K instead of χ yields

$$t = \frac{2}{(b-2)A(Y\sigma)^b}\left[\left(\frac{K_i}{Y\sigma}\right)^{2-b} - \left(\frac{K_f}{Y\sigma}\right)^{2-b}\right]$$

$$\approx \frac{2K_i^{2-b}}{(b-2)A(Y\sigma)^b} \qquad \text{if } K_i^{b-2} \ll K_f^{b-2} \quad \text{or} \quad K_i^{2-b} \gg K_f^{2-b}$$

2-34. Substituting Eq. (2-82) into Eq. (2-86) gives

$$\frac{d\chi}{dt} = AK^b = AY^b\chi^{b/2}\sigma^b$$

Integrating this from χ_i to χ_p where

$$\chi_i = \left(\frac{K}{Y\sigma_i}\right)^2 \quad \text{and} \quad \chi_p = \left(\frac{K}{Y\sigma_p}\right)^2$$

are the initial crack depth and the crack depth after proof testing, respectively, yields

$$\int_{\chi_i}^{\chi_p} \chi^{-b/2}\, d\chi = AY^b \int_0^{t_p} \sigma^b\, dt$$

or

$$\frac{1}{1-\frac{b}{2}} \left[\chi_p^{1-b/2} - \chi_i^{1-b/2}\right] = AY^b \sigma_p^b\, t_p$$

for a constant stress σ_p. Substituting for χ_i and χ_p gives

$$\left(\frac{2}{b-2}\right)\left(\frac{K}{Y}\right)^{2-b} \left[\sigma_i^{b-2} - \sigma_p^{b-2}\right] = AY^b \sigma_p^b\, t_p$$

or

$$\left(\frac{2}{b-2}\right)\left(\frac{K}{Y}\right)^{2-b} \frac{1}{AY^b}\left[\sigma_i^{b-2} - \sigma_p^{b-2}\right] = B\left[\sigma_i^{b-2} - \sigma_p^{b-2}\right] = \sigma_p^b\, t_p$$

which is Eq. (2-87).

When a static stress σ_s is applied after proof testing, the time to failure is found from Eq. (2-86):

$$\int_{\chi_p}^{\chi_s} \chi^{-b/2}\, d\chi = AY^b \sigma_s^b \int_0^{t_s} dt$$

where χ_s is the crack depth at the fiber failure point. Integrating (as above) we get Eq. (2-89):

$$B\left[\sigma_p^{b-2} - \sigma_s^{b-2}\right] = \sigma_s^b\, t_s$$

Adding Eqs. (2-87) and (2-89) yields Eq. (2-90).

2-35. (a) Substituting N_S as given by Eq. (2-92) and N_p as given by Eq. (2-93) into Eq. (2-94) yields

$$F = 1 - \exp\left(-\frac{L}{L_0}\left\{ \frac{\left[\left(\sigma_p^b t_p + \sigma_s^b t_s\right)/B + \sigma_s^{b-2}\right]^{\frac{m}{b-2}}}{\sigma_0^m} - \frac{\left(\sigma_p^b t_p / B + \sigma_p^{b-2}\right)^{\frac{m}{b-2}}}{\sigma_0^m} \right\}\right)$$

$$= 1 - \exp\left(-\frac{L}{L_0\sigma_0^m}\left[\sigma_p^b t_p / B + \sigma_p^{b-2}\right]^{\frac{m}{b-2}} \left\{ \left[\frac{\left(\frac{\sigma_p^b t_p + \sigma_s^b t_s}{B}\right) + \sigma_s^{b-2}}{\sigma_p^b t_p / B + \sigma_p^{b-2}} \right]^{\frac{m}{b-2}} - 1 \right\} \right)$$

$$= 1 - \exp\left(-LN_p \left\{ \frac{\left[1 + \frac{\sigma_s^b t_s}{\sigma_p^b t_p} + \left(\frac{\sigma_s}{\sigma_p}\right)^b \frac{B}{\sigma_s^2 t_p} \right]^{\frac{m}{b-2}}}{1 + \frac{B}{\sigma_p^2 t_p}} - 1 \right\} \right)$$

$$\approx 1 - \exp\left(-LN_p \left\{ \left[\left(1 + \frac{\sigma_s^b t_s}{\sigma_p^b t_p}\right) \frac{1}{1 + \frac{B}{\sigma_p^2 t_p}} \right]^{\frac{m}{b-2}} - 1 \right\} \right)$$

(b) For the term given by Eq. (2-96) we have

$$\left(\frac{\sigma_s}{\sigma_p}\right)^b \frac{B}{\sigma_s^2 t_p} = (0.3)^{15} \; \frac{0.5 \;(\mathrm{MN/m^2})^2 \; \mathrm{s}}{\left[0.3 \;(350 \;\mathrm{MN/m^2})\right]^2 10 \;\mathrm{s}} = 6.5\times 10^{-14}$$

Thus this term can be neglected.

2-36. The failure probability is given by Eq. (2-85). For equal failure probabilities of the two fiber samples, $F_1 = F_2$, or

$$1 - \exp\left[-\left(\frac{\sigma_{1c}}{\sigma_0}\right)^m \frac{L_1}{L_0}\right] = 1 - \exp\left[-\left(\frac{\sigma_{2c}}{\sigma_0}\right)^m \frac{L_2}{L_0}\right]$$

which implies that

$$\left(\frac{\sigma_{1c}}{\sigma_0}\right)^m \frac{L_1}{L_0} = \left(\frac{\sigma_{2c}}{\sigma_0}\right)^m \frac{L_2}{L_0}$$

or

$$\frac{\sigma_{1c}}{\sigma_{2c}} = \left(\frac{L_2}{L_1}\right)^{1/m}$$

If L_1 = 20 m, then σ_{1c}= 4.8 GN/m^2

If L_2 = 1 km, then σ_{2c}= 3.9 GN/m^2

Thus

$$\left(\frac{4.8}{3.9}\right)^m = \frac{1000}{20} = 50$$

gives

$$m = \frac{\log 50}{\log(4.8/3.9)} = 18.8$$