Question 1. Random Process (25 marks)

Two random processes x(t) and y(t) are:

$$x(t) = A\cos(\omega_0 t + \varphi) \text{ and } y(t) = B\cos(n\omega_0 t + n\varphi)$$

where n is an integer not equal to 1 and A and B are constants. φ is a uniformly distributed random variable in the range (0, 2π). Show that the two processes are incoherent.

Proof: $$m_{12} = \overline{x_1 x_2} = \int_0^{2\pi} A\cos(\omega_0 t + \varphi) B\cos(n\omega_0 t + n\varphi) \frac{1}{2\pi} d\varphi$$

$$= \frac{AB}{4\pi} \int_0^{2\pi} \left\{ \cos\left[(n-1)\omega_0 t + (n-1)\varphi\right] + \cos\left[(n+1)\omega_0 t + (n+1)\varphi\right] \right\} d\varphi$$

$$\because \quad n \neq 1$$

$$\therefore \quad m_{12} = 0$$

And $$\overline{x_1} = \int_0^{2\pi} A\cos(\omega_0 t + \varphi) \frac{1}{2\pi} d\varphi = 0$$

$$\overline{x_2} = \int_0^{2\pi} B\cos(n\omega_0 t + n\varphi) \frac{1}{2\pi} d\varphi = 0$$

$$\therefore m_{12} = \overline{x_1 x_2} = \overline{x_1}\,\overline{x_2} = m_1 m_2 = 0$$

$\therefore$ two processes are incoherent

Question 2. Bandpass White Noise (25 marks)

The white Gaussian noise process $N(t)$ with power spectrum $N_0/2$ passes through an ideal bandpass filter with frequency response

$$H(f) = \begin{cases} 1 & |f - f_0| < W \\ 0 & \text{otherwise} \end{cases}$$

Where $W \ll f_0$. The output process is denoted by $X(t)$. Find the power spectrum and the cross-spectral density of the in-phase and quadrature components in the following two cases:

1. f_c is chosen to be equal to f_0
2. f_c is chosen to be equal to $f_0 - W$

(1) The process $X(t)$ is obviously a bandpass process whose power-spectral density is given by

$$S_X(f) = \begin{cases} \frac{N_0}{2}, & |f - f_c| < W, \\ 0 & \text{otherwise.} \end{cases}$$

If we choose f_c as the central frequency, then to obtain power spectra of $X_c(t)$ and $X_s(t)$ we have to shift the positive-frequency part of $S_X(f)$ to the left by $f_0 = f_c$ and the negative-frequency part of it to the right by the same amount and add the results. If we do this, the resulting spectrum will be

$$S_{X_c}(f) = S_{X_s}(f) = \begin{cases} N_0, & |f| < W, \\ 0 & \text{otherwise.} \end{cases}$$

Since we are choosing f_o to be the axis of symmetry of the spectrum of $X(t)$, the process $X_c(t)$ and $X_s(t)$ will be independent with this choice and

$$S_{X_c X_s}(f) = 0$$

(2) If we choose $f_0 = f_c - W$ as the central frequency, the result will be different.

$$S_{X_c}(f) = S_{X_s}(f) = \begin{cases} \frac{N_0}{2}, & |f| < 2W, \\ 0, & \text{otherwise.} \end{cases}$$

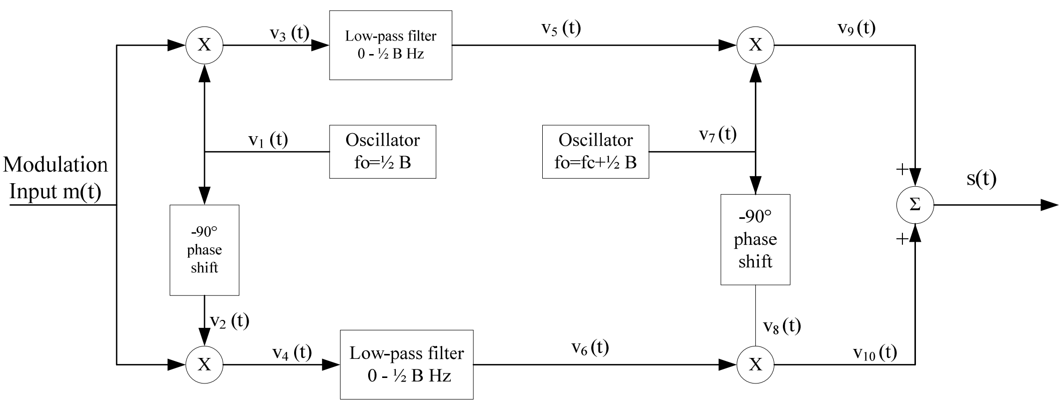
and for the cross-spectral density, we will have

$$S_{X_c}(f) = S_{X_s}(f) = \begin{cases} -j\frac{N_0}{2}, & -2W < f < 0 \\ j\frac{N_0}{2}, & 0 < f < 2W \\ 0, & \text{otherwise.} \end{cases}$$

Question 3.SSB Generation (25 marks)

SSB signals can be generated by the Weaver's method as shown in figure, where B is the bandwidth of m(t),

1. Find a mathematical expression that describes the waveform out of each block on the block diagram.
2. Show that s(t) is an SSB signal.



Solution:

$$v_3(t) = m(t)\cos\omega_b t$$

$$v_5(t) = \frac{1}{2}\{m(t)\cos\omega_b t + \hat{m}(t)\sin\omega_b t\}$$

$$v_9(t) = \frac{1}{2}\{m(t)\cos\omega_b t + \hat{m}(t)\sin\omega_b t\} \times \cos[(\omega_c + \omega_b)t]$$

$$v_4(t) = m(t)\cos(\omega_b t - \frac{\pi}{2}) = m(t)\sin\omega_b t$$

$$v_6(t) = \frac{1}{2}\{m(t)\cos(\omega_b t - \frac{\pi}{2}) + \hat{m}(t)\sin(\omega_b t - \frac{\pi}{2})\}$$

$$= \frac{1}{2}\{m(t)\sin\omega_b t - \hat{m}(t)\cos\omega_b t\}$$

$$v_{10}(t) = \frac{1}{2}\{m(t)\sin\omega_b t - \hat{m}(t)\cos\omega_b t\} \times \sin[(\omega_c + \omega_b)t]$$

$$
\begin{aligned}
s(t) &= v_9(t) + v_{10}(t) \\
&= \frac{1}{2}\{m(t)\cos\omega_b t + \hat{m}(t)\sin\omega_b t\}\cos\omega_d t + \frac{1}{2}\{m(t)\sin\omega_b t - \hat{m}(t)\cos\omega_b t\}\sin\omega_d(t) \\
&= \frac{1}{2}m(t)\{\cos\omega_b t\cos\omega_d t + \sin\omega_b t\sin\omega_d t\} + \frac{1}{2}\hat{m}(t)\{\sin\omega_b t\cos\omega_d t - \cos\omega_b t\sin\omega_d t\} \\
&= \frac{1}{2}m(t)\cos[(\omega_d - \omega_b)t] - \frac{1}{2}\hat{m}(t)\sin[(\omega_d - \omega_b)t] \\
&= \frac{1}{2}\{m(t)\cos\omega_c t - \hat{m}(t)\sin\omega_c t\}
\end{aligned}
$$

$$\omega_b = B/2 \qquad \omega_d = \omega_c + \omega_b$$

It is a USSB signal.

Question 4. FM System (25 marks)

Design an FM system that achieves an SNR at the receiver equal to 40 dB and requires the minimum amount of transmitter power. The bandwidth of the channel is 120 KHz, the message bandwidth is 10 KHz, the average-to-peak-power ratio for the message is 1/2, and the one-sided noise power-spectral density is 10^{-8} W/Hz. What is the required transmitter power if the signal is attenuated by 40 dB in the transmission through the channel?

First, by Carson's rule,

$$B_c = 2(\beta + 1)W$$

$$120000 = 2(\beta + 1) \times 10000$$

from which we obtain $\beta = 5$. Using the relation

$$\left(\frac{S}{N}\right)_o = 60\beta^2(\beta + 1)P_{M_n}$$

where, $P_{M_n} = \frac{1}{2}$, with $\left(\frac{S}{N}\right)_o = 10^4$, we obtain $\beta \approx 6.6$. Since the value of β given by the bandwidth constraint is less than the value of β given by the power constraint, we limited in bandwidth. Therefore, we choose $\beta = 5$. From this relation,

$$\left(\frac{S}{N}\right)_o = \frac{3}{2}\beta^2\left(\frac{S}{N}\right)_b$$

we can get

$$\left(\frac{S}{N}\right)_b = \frac{800}{3} \approx 24.46dB$$

Since $\left(\frac{S}{N}\right)_b = \frac{P_R}{N_o W}$ with W=10,000 and $N_o = 10^{-8}$

$$P_R = \frac{8}{300} \approx -15.74dB$$

$$P_T = -15.74 + 40 = 24.26dB \approx 266.66W$$