Signals and Spectra (4)

6-4, 6-5, 6-6

Lecture 6, 2008-09-23

Contents

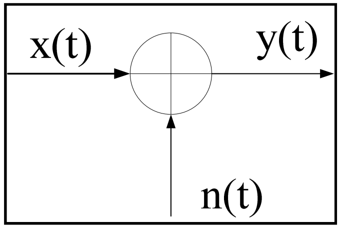
- White Noise
- Linear Systems and Random Process
- Bandwidth Measures
- Gaussian Process

White-Noise Processes

- A random process x(t) is said to be a white-noise process if the PSD is constant over all frequencies, $P_x(f) = N_0/2$, where N_0 is a positive constant.
- The autocorrelation function for the white noise is obtained by taking the inverse Fourier transform $R_x(t) = N_0/2\ \delta(t)$
- Any two different samples of a white noise process are uncorrelated. Since thermal noise is a Gaussian process and the sample are uncorrelated, the noise samples are also independent.
- The effect on the detection process of a channel with Additive White Gaussian Noise is that the noise affects each transmitted symbol independently. Memoryless channel.

Noise in communication systems

- AWGN: additive white Gaussian noise
  - Additive: Noise is added (not multiplied) to the signal
  - White: has constant PSD (equal power for all frequency)
  - Gaussian: in every time-instant (sampling instant), the noise is Gaussian random variable
- Noise is usually assumed zero-mean AWGN



Signal model: $y(t) = x(t) + n(t)$

zero-mean AWGN $n(t)$ properties:

i) PSD: $G_n(f) = \frac{N_0}{2}$ watts/Hz

ii) Autocorrelation: $R_n(\tau) = \frac{N_0}{2}\delta(\tau)$

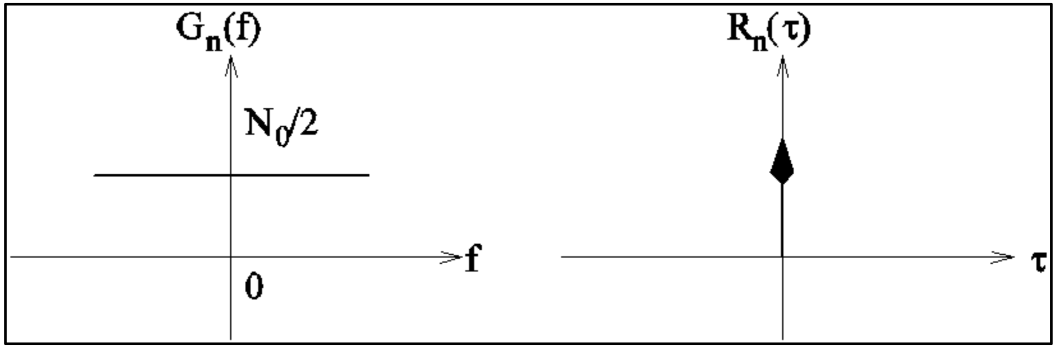
iii) pdf: $p(n) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{n^2}{2\sigma^2}}$

Cont'

- AWGN is a useful abstract noise model, although it is not practical due to infinite power
- In sampled process (discrete process), since δ(0)=1, we still have

$$\sigma^2 = E\{X^2\} = \frac{N_0}{2}$$

  - Discrete zero-mean AWGN: power & variance are both $N_0/2$



AWGN PSD & Auto-correlation

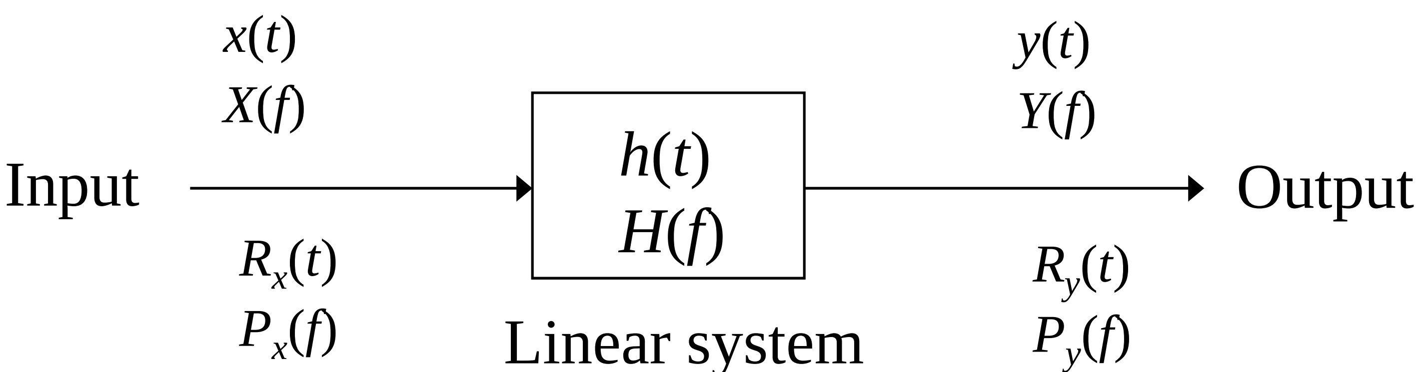
Input-Output Relationships

- A linear time-invariant system may be described by its impulse response h(t) or equivalently, by its transfer function H(f).
- Deterministic signals

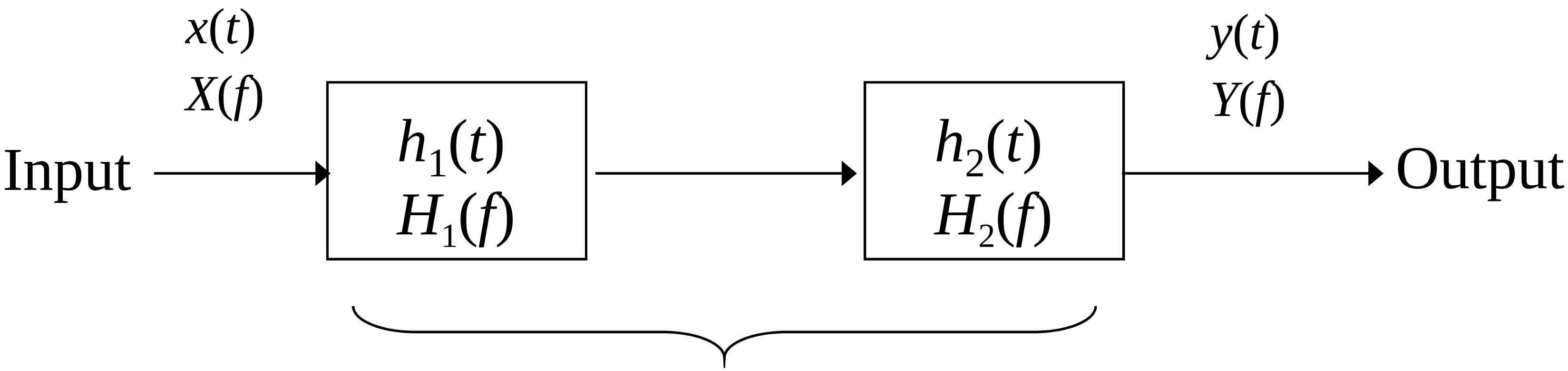
$$y(t) = h(t)*x(t)$$
$$Y(f) = H(f)X(f)$$

- If a wide-sense stationary random process x(t) is applied to the input of a time-invariant linear network with impulse response h(t), the output autocorrelation is

$$R_y(t) = h(-t)*h(t)*R_x(t)$$
$$P_y(f) = |H(f)|^2 P_X(f)$$



Two Linear Cascaded Networks



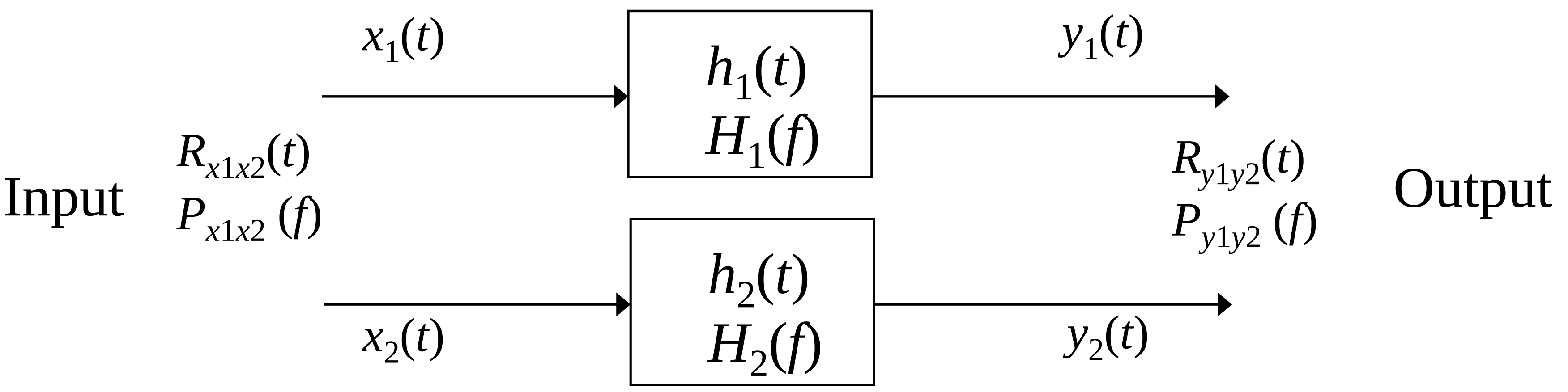
Overall response: $h(t) = h_1(t) * h_2(t)$

$$H(f) = H_1(f)\, H_2(f)$$

Two Linear Systems

- Let x1(t) and x2(t) be wide-sense stationary inputs for two time-invariant linear systems, then the output cross-correlation function is

$$R_{y1y2}(t) = h_1(-t) * h_2(t) * R_{x1x2}(t)$$
$$P_{y1y2}(f) = H_1^*(f)\, H_2(f)\, P_{X1x2}(f)$$



Two Linear Systems

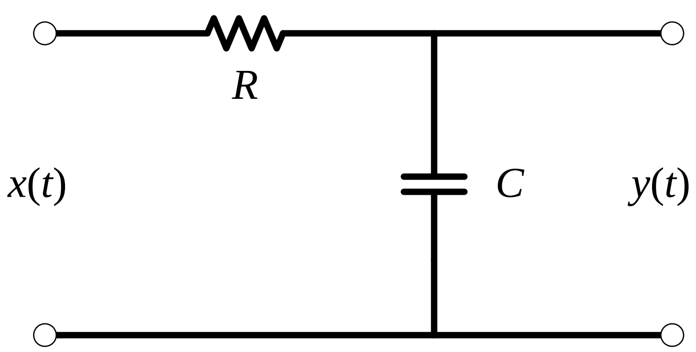
Example

- Output autocorrelation and PSD for an RC low-pass filter, white noise

$$P_x(f) = \frac{N_0}{2}$$

$$P_y(f) = \left|H(f)\right|^2 P_x(f) = \frac{N_0/2}{1+\left(f/B_{3dB}\right)^2}$$

$$R_y(\tau) = \frac{N_0}{4RC} e^{-|\tau|/(RC)}$$



$$H(f) = \frac{1}{1+j\left(\dfrac{f}{B_{3dB}}\right)}, \quad \text{where } B_{3dB} = \frac{1}{2\pi RC}$$

Example

- Signal-to-noise ratio at the output of an RC low-pass filter

Input signal $$x(t) = s_i(t) + n_i(t)$$

$$s_i(t) = A_0 \cos(\omega_0 t + \theta_0)$$

$$P_{n_i}(f) = N_0 / 2$$

Input power

$$\left\langle s_i^2(t) \right\rangle = A_0^2 / 2$$

$$\left\langle n_i^2 \right\rangle = \int_{-\infty}^{\infty} P_{n_i}(f) df = \infty$$

Input SNR $$\left(\frac{S}{N} \right)_{in} = \frac{\left\langle s_i^2(t) \right\rangle}{\left\langle n_i^2 \right\rangle} = 0$$

Cont'

Output signal

$$y(t) = s_o(t) + n_o(t)$$

$$s_o(t) = s_i(t) * h(t)$$

Output power

$$\left\langle s_o^2(t) \right\rangle = A_0^2 \left| H(f_0) \right|^2 / 2$$

$$\left\langle n_o^2 \right\rangle = N_0 / (4RC)$$

Output SNR

$$\left(\frac{S}{N} \right)_{out} = \frac{\left\langle s_0^2(t) \right\rangle}{\left\langle n_0^2 \right\rangle} = \frac{2A_0^2 \left| H(f_0) \right|^2 RC}{N_0} = \frac{2A_0^2 RC}{N_0 \left[1 + (2\pi f_0 RC)^2 \right]}$$

Bandwidth Measures

- Equivalent Bandwidth

  For a wide-sense stationary process x(t), the equivalent bandwidth is

$$B_{eq} = \frac{1}{P_x(f_0)} \int_0^{\infty} P_x(f)df = \frac{R_x(0)}{2P_x(f_0)}$$

  Where f_0 is the frequency at which $P_x(f)$ is a maximum

- RMS Bandwidth

  If x(t) is a low-pass wide-sense stationary process, the rms bandwidth is

$$B_{rms} = \sqrt{\overline{f^2}}$$

$$\overline{f^2} = \int_{-\infty}^{\infty} f^2 \left[\frac{P_x(f)}{\int_{-\infty}^{\infty} P_x(\lambda)d\lambda} \right] df = \frac{\int_{-\infty}^{\infty} f^2 P_x(f)df}{\int_{-\infty}^{\infty} P_x(\lambda)d\lambda}$$

Theorem

- For a wide-sense stationary process x(t), the mean-squared frequency is

$$\overline{f^2} = \left[-\frac{1}{(2\pi)^2 R_x(0)} \right] \frac{d^2 R_x(\tau)}{d\tau^2} \bigg|_{\tau=0}$$

- Proof

Example

- Equivalent bandwidth and RMS bandwidth for an RC LRF

$$B_{eq} = \frac{P_x(0)}{2P_x(f_0)} = \frac{N_0/(4RC)}{2(N_0/2)} = \frac{1}{4RC} = \frac{\pi B_{3dB}}{2}$$

$$B_{rms} = \sqrt{\frac{\int_{-\infty}^{\infty} f^2 P_y(f)df}{R_y(0)}} = \sqrt{\frac{1}{2\pi^2 RC}\int_{-\infty}^{\infty} \frac{f^2}{(B_{3dB})^2 + f^2}df}$$

The Gaussian Random Process

- A random process x(t) is said to be Gaussian if the random variable $x_1=x(t_1)$, $x_2=x(t_2)$, …, $x_N=x(t_N)$ have an N-dimensional Gaussian PDF for any N and $t_1, t_2, \ldots, t_N$

  Let x be the column vector denoting the N random variables:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{bmatrix} = \begin{bmatrix} x(t_1) \\ x(t_2) \\ \cdots \\ x(t_N) \end{bmatrix}$$

The N-dimensional Gaussian PDF is

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{(2\pi)^{N/2} \left|\text{Det } \mathbf{C}\right|^{1/2}} e^{-\left[(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x}-\mathbf{m})\right]/2}$$

Cont’

Where the mean vector is

$$\mathbf{m} = \bar{\mathbf{x}} = \begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \ldots \\ \overline{x_N} \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ \ldots \\ m_N \end{bmatrix}$$

The covariance matrix is

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1N} \\ c_{21} & c_{22} & \ldots & c_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ c_{11} & c_{N2} & \ldots & c_{NN} \end{bmatrix}$$

Where

$$c_{ij} = \overline{(x_i - m_i)(x_j - m_j)} = \overline{[x(t_i) - m_i][x(t_j) - m_j]}$$

Properties of Gaussian Processes

- $f_x(\mathbf{x})$ depends only on **C** and **m**, which is another way of saying that the N-dimensional Gaussian PDF is completely specified by the first- and second-order moments
- Since the $\{x_i = x(t_i)\}$ are jointly Gaussian, the $x_i = x(t_i)$ are individually Gaussian.
- When **C** is a diagonal matrix, the random variables are uncorrelated. Furthermore, the Gaussian random variables are independent when they are uncorrelated.
- A linear transformation of a set of Gaussian random variables products another set of Gaussian random variables
- A wide-sense stationary Gaussian process is also strict-sense stationary

Theorem

- If the input to a linear system is a Gaussian random process, the system output is also a Gaussian process.
- Proof

Example

- White Gaussian-Noise Process input to the LPF, the first order pdf (Homework)
- The output is nonwhite, but Gaussian

Homework

- 6-21, 6-25, 6-27, 6-33, 6-34
- Show the first order pdf of AWGN after an LPF.