Signals and Spectra (3)

6-1, 6-2

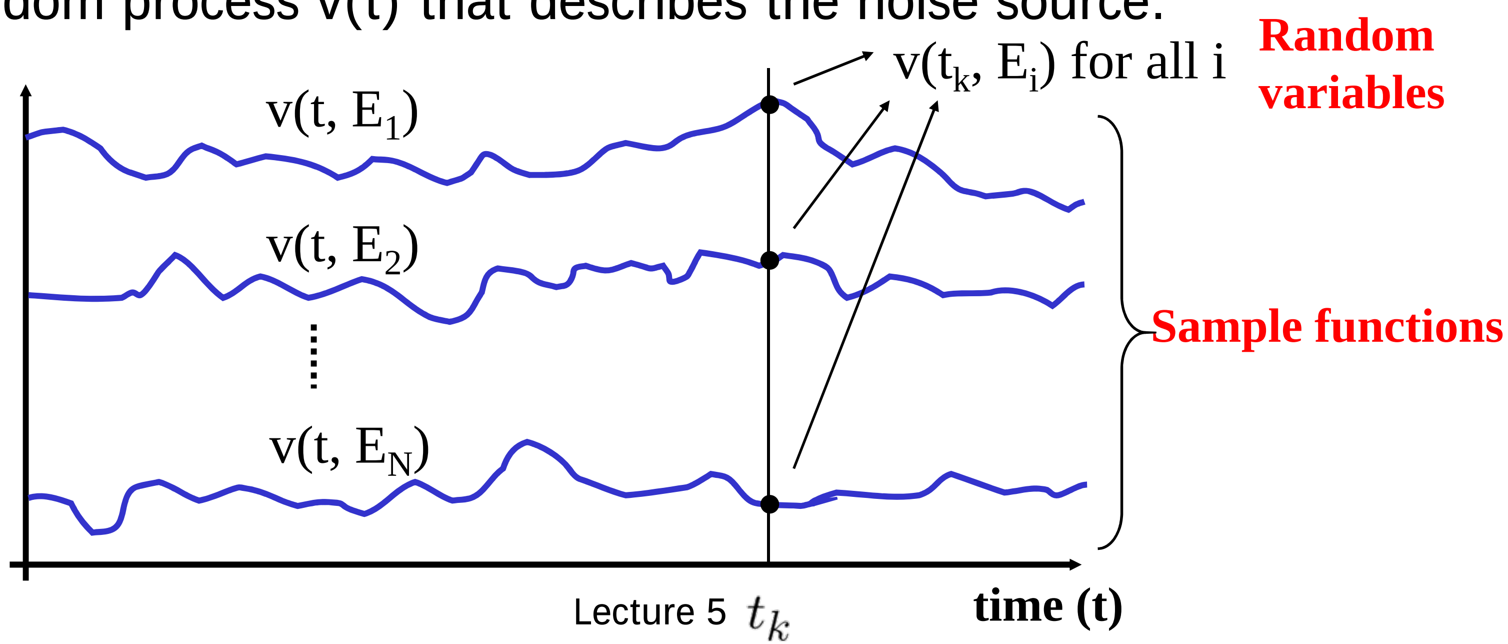
Lecture 5, 2008-09-19

Introduction

- Random Process
- Stationary
- Ergodic
- Correlation Functions
- Power Spectral Density
- Wiener-Khintchine Theorem

Random Process

- A real random process (or stochastic process) is an indexed set of real functions of some parameter (usually time) that has certain statistical properties.
- Imagine voltage waveforms that might be emitted from a noise source. In general, $v(t, E_i)$ denotes the waveform that is obtained when the event Ei of the sample space occurs. $v(t, E_i)$ is said to be a sample function of the sample space. The set of all possible sample functions $\{v(t, E_i)\}$ is called the ensemble and defines the random process v(t) that describes the noise source.



Random Process

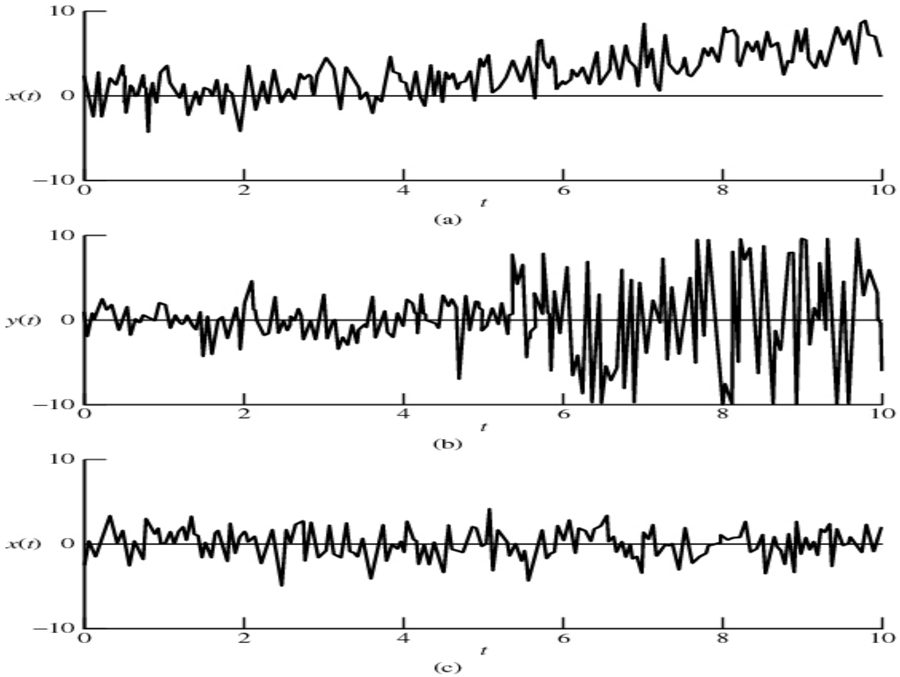
- The difference between a random variable and a random process is that for a random variable, an outcome in the sample space is mapped into a number, whereas for a random process it is mapped into a function of time.
- A random process may be described by an indexed set of random variables.
- To describe a general random process x(t) completely, an N-dimensional PDF, $f_x(x)$, is required, where $X = (x_1, x_2, ..., x_N)$, $x_j = x(t_j)$ and $N \to \infty$. $f_X(\boldsymbol{x}) = f_X(x(t_1), x(t_2),..., x(t_N))$

Stationary

- A random process x(t) is said to be stationary to the order N if, for any $t_1, t_2, \ldots, t_N$

  $$f_x(x(t_1), x(t_2), \ldots, x(t_N)) = f_x(x(t_1+t_0), x(t_2+t_0), \ldots, x(t_N+t_0))$$

  Where t_0 is any arbitrary real constant. Furthermore, the process is said to be strictly stationary if it is stationary to the order $N \rightarrow \infty$.



(a) Time-varying mean

(b) Time-varying variance

(c) Stationary

Example

- First-order stationary

  The requirement for first-order stationary is that is the first-order PDF not be a function of time. Let the random process be $x(t)=A\sin(\omega t+\theta)$

  Case 1: Stationary Result. Assume A and ω are deterministic constants and θ is a random variable uniformly distributed over $-\pi$ to π.

$$f(x)=\begin{cases}\dfrac{1}{\pi\sqrt{A^2-x^2}}, |x|\le A \\ 0, |x|>A\end{cases}$$

  Case 2: Nonstationary Result. Assume A, ω and θ are deterministic constants.

$$f(x)=\delta(x-A\sin(\omega t+\theta))$$

Ergodic

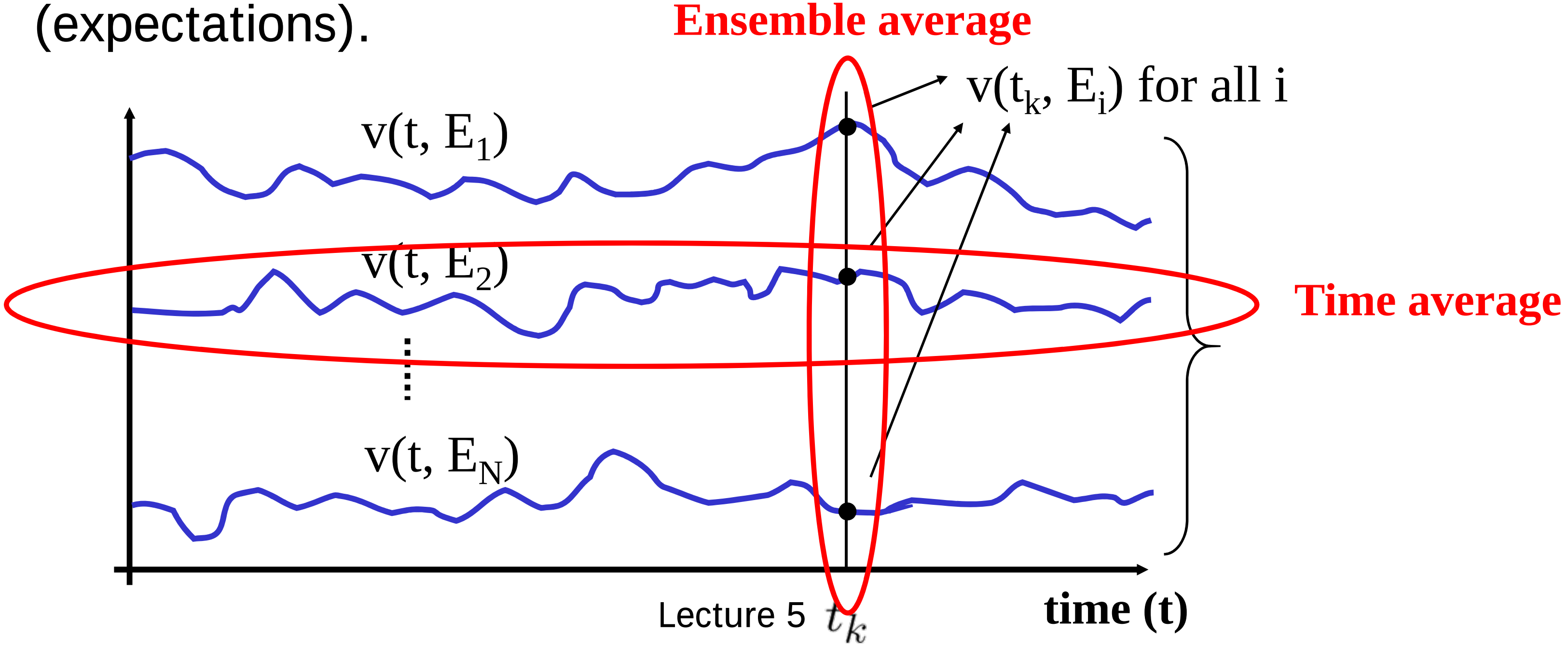
- The time average is

$$\langle [x(t)] \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [x(t)]\, dt$$

- The ensemble average is

$$\overline{[x(t)]} = \int_{-\infty}^{\infty} [x]\, f_x(x) dx = m_x$$

- A random process is said to be ergodic if all time averages of any sample function are the corresponding ensemble averages (expectations).



Example

- Let a random process be given by $x(t)=A\cos(\omega t+\theta)$, Assume A and ω are deterministic constants and θ is a random variable uniformly distributed over 0 to 2π.
- The ensemble averages

$$\overline{x} = \int_{-\infty}^{\infty} \left[x(\theta)\right] f_\theta(\theta) d\theta = \int_0^{2\pi} \left[A\cos(\omega t + \theta)\right] \frac{1}{2\pi} d\theta = 0$$

- The time averages

$$\left\langle x(t) \right\rangle = \frac{1}{T_0} \int_0^{T_0} A\cos(\omega t + \theta) dt = 0$$

- The time average is equal to the ensemble average.
- Shall we conclude that the process is ergodic?
- No. Because we have not evaluated all the possible time and ensemble averages. In general, it is difficult to prove that a process is ergodic

Ergodic vs. Stationary

- The ergodic process must be stationary, because otherwise the ensemble averages would be a function of time.
- However, if a process is known to be stationary, it may or may not be ergodic.
- Excise 6.2

Correlation Functions

- The autocorrelation function of a random process x(t) is

$$R_x(t_1,t_2)=\overline{x(t_1)x(t_2)}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1x_2\,f_x(x_1,x_2)\,dx_1dx_2$$

  where $x_1=x(t_1)$ and $x_2=x(t_2)$.

- If the process is stationary to the second order, the autocorrelation function is a function only of the time difference $\tau= t_2- t_1$

$$R_x(\tau)=\overline{x(t)x(t+\tau)}$$

  if x(t) is second-order stationary

- A random process is said to be wide-sense stationary if

$$\frac{d\,\overline{x(t)}}{dt}=0 \text{ and } R_x(t_1,t_2)=R_x(t_2-t_1)$$

- The auto-correlation function has following properties:

$$R_x(0)=\overline{x^2(t)}$$

$$R_x(\tau)=R_x(-\tau)$$

$$R_x(0)\geq\left|R_x(\tau)\right|$$

Cross-Correlation Function

- The cross-correlation function of two random process x(t) and y(t) is

$$R_{xy}(t_1,t_2) = \overline{x(t_1)y(t_2)}$$

- If x(t) and y(t) are jointly stationary, the cross-correlation function is

$$R_{xy}(t_1,t_2) = R_{xy}(t_2 - t_1)$$

- The cross-correlation function has following properties:

$$R_{xy}(-\tau) = R_{yx}(\tau)$$

$$\left|R_{xy}(\tau)\right| \le \sqrt{R_x(0)R_y(0)}$$

$$\left|R_{xy}(\tau)\right| \le \frac{1}{2}\left[R_x(0) + R_y(0)\right]$$

- Two random processes x(t) and y(t) are said to be uncorrelated if

$$R_{xy}(\tau) = \left[\overline{x(t)}\right]\left[\overline{y(t+\tau)}\right] = m_x m_y \qquad \text{for all values of } \tau$$

- Two random processes x(t) and y(t) are said to be orthogonal if

$$R_{xy}(\tau) = 0 \qquad \text{for all values of } \tau$$

Correlation Measurement

Power Spectral Density

- The power spectral density for a random process x(t) is given by

$$P_x(f) = \lim_{T\to\infty}\left(\frac{\overline{\left[|X_T(f)|^2\right]}}{T}\right)$$

where

$$X_T(f) = \int_{-T/2}^{T/2} x(t) e^{-j2\pi ft} dt$$

Wiener-Khintchine Theorem

- When x(t) is a wide-sense stationary process, the PSD can be obtained from the Fourier transform of the autocorrelation function

$$P_x(f) = \int_{-\infty}^{\infty} R_x(t) e^{-j2\pi ft} dt$$

Conversely,

$$R_x(t) = \int_{-\infty}^{\infty} P_x(f) e^{j2\pi ft} dt$$

- To calculate the PSD of a random process:

1. Direct method: Use the original definition of PSD
2. Indirect method: Use Wiener-Khintchine Theorem

Proof of W-K Theorem

From the definition of PSD

$$P_x(f) = \lim_{T\to\infty}\left(\frac{\overline{\left[|X_T(f)|^2\right]}}{T}\right)$$

Where

$$\overline{|X_T(f)|^2} = \overline{\left|\int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt\right|^2}$$

$$= \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} \overline{x(t_1)x(t_2)}e^{-j2\pi f(t_1-t_2)}dt_1dt_2$$

Proof of W-K Theorem

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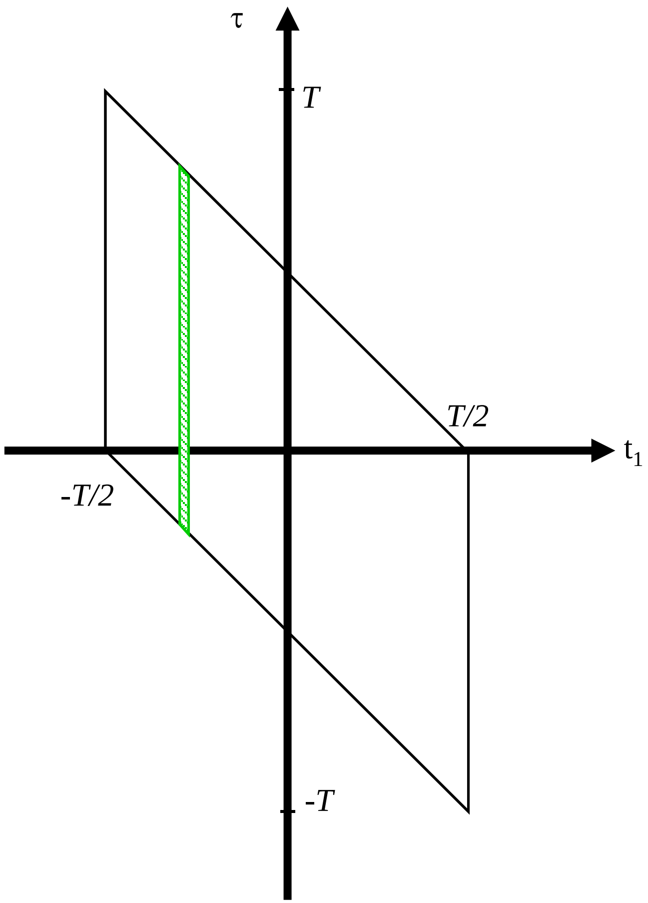
$$= \int_{-T/2}^{T/2}\left[\int_{-T/2-t_1}^{T/2-t_1} R_x(t_1, t_1+\tau)e^{-j2\pi f\tau}d\tau\right]dt_1$$



Region of Integration

Proof of W-K Theorem

From the definition of PSD

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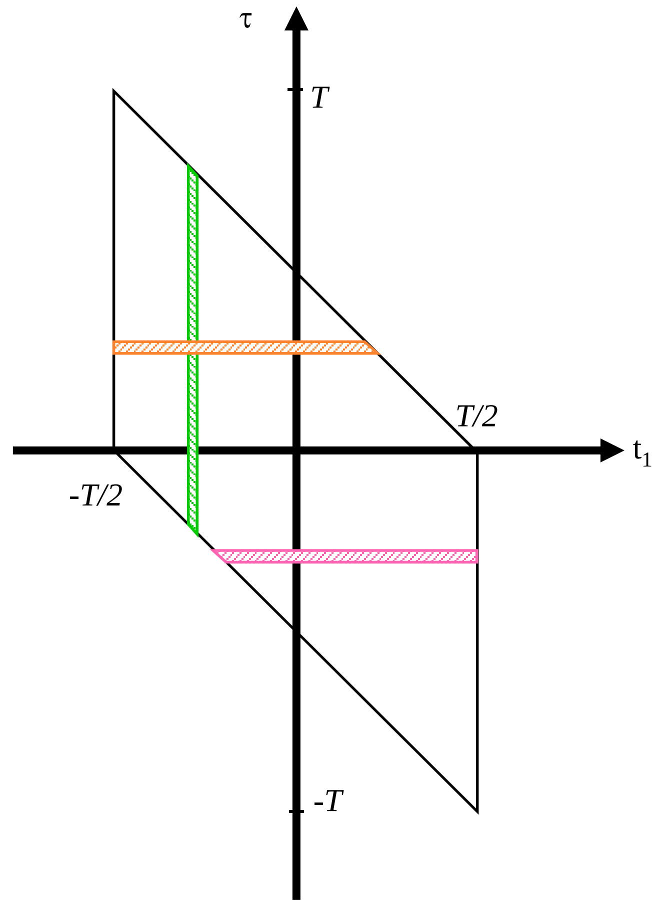
$$= \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} \overline{x(t_1)x(t_2)}e^{-j2\pi f(t_1-t_2)}dt_1dt_2$$

$$= \int_{-T/2}^{T/2}\left[\int_{-T/2-t_1}^{T/2-t_1} R_x(t_1,t_1+\tau)e^{-j2\pi f\tau}d\tau\right]dt_1$$

$$= \int_{-T}^{0}\left[\int_{-T/2-\tau}^{T/2} R_x(t_1,t_1+\tau)e^{-j2\pi f\tau}dt_1\right]d\tau$$

$$+\int_{0}^{T}\left[\int_{-T/2}^{T/2-\tau} R_x(t_1,t_1+\tau)e^{-j2\pi f\tau}dt_1\right]d\tau$$



Region of Integration

Cont'

If x(t) is stationary,

$$\overline{|X_T(f)|^2} = \int_{-T}^{0} R_x(\tau)e^{-j2\pi f\tau}\left[t_1\Big|_{-T/2-\tau}^{T/2}\right]d\tau + \int_{0}^{T} R_x(\tau)e^{-j2\pi f\tau}\left[t_1\Big|_{-T/2}^{T/2-\tau}\right]d\tau$$

$$= \int_{-T}^{T}\left(T-|\tau|\right)R_x(\tau)e^{-j2\pi f\tau}d\tau$$

Then

$$P_x(f) = \lim_{T\to\infty}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)R_x(\tau)e^{-j2\pi f\tau}d\tau$$

$$= \int_{-\infty}^{\infty} R_x(\tau)e^{-j2\pi f\tau}d\tau$$

Example

- example 6-3

Properties of PSD

- $P_x(f)$ is always real
- $P_x(f) \geq 0$
- When x(t) is real, $P_x(-f) = P_x(f)$
- Total normalized power $\int_{\infty}^{\infty} P_x(f)\,df = P$
- $P_x(0) = \int_{-\infty}^{\infty} R_x(\tau)\,d\tau$

General Formula for the PSD of Digital Signals

$$P_x(f) = \frac{|F(f)|^2}{T_s}\left[\sum_{-\infty}^{\infty} R(k) e^{jk\omega T_s}\right]$$

$$R(k) = \overline{a_n a_{n+k}} = \sum_{i=1}^{I} (a_n a_{n+k})_i P_i$$

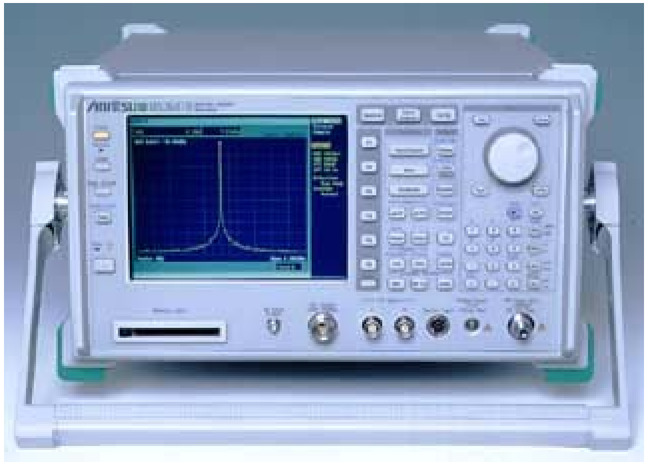
- Pi is the probability of getting the product $(a_n a_{n+k})_I$, of which there are I possible values
- F(f) is the spectrum of the pulse shape of the digital symbol

White-Noise Processes

- A random process x(t) is said to be a white-noise process if the PSD is constant over all frequencies, $P_x(f) = N_0/2$, where N_0 is a positive constant.
- The autocorrelation function for the white noise is obtained by taking the inverse Fourier transform $R_x(t) = N_0/2\ \delta(t)$
- Any two different samples of a white noise process are uncorrelated. Since thermal noise is a Gaussian process and the sample are uncorrelated, the noise samples are also independent.
- The effect on the detection process of a channel with Additive White Gaussian Noise is that the noise affects each transmitted symbol independently. Memoryless channel.

Measurement of PSD

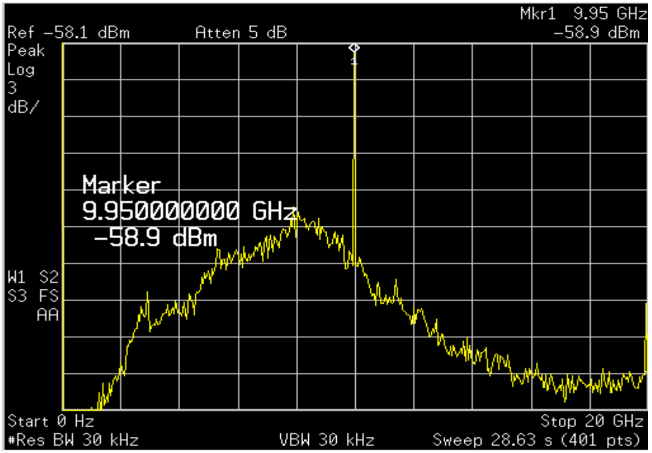
- Analog techniques
  - Narrowband filters
  - RF spectrum analyzer
- Numerical computation



$$P_T(f) = \frac{|X_T(f)|^2}{T}$$

$$\overline{P_T(f)} = \overline{\left[\frac{|X_T(f)|^2}{T}\right]} = F\left[R_x(\tau)\Lambda\left(\frac{\tau}{T}\right)\right]$$

$$= TP_x(f) * \left(\frac{\sin \pi fT}{\pi fT}\right)^2$$



Homework

- 6-2, 6-3, 6-9, 6-15, 6-17