Signals and Spectra (1)

2-1, 2-7, 2-9

Lecture 3, 2008-09-12

Introduction

- Practical Waveforms
- Time Average Operators
- Bandlimited Signals
- Bandwidth

Properties of Signals and Noise

- In communication systems, the received waveform is usually categorized into the desired part containing the information and the extraneous or undesired part. The desired part is called the signal, and the undesired part is called noise.

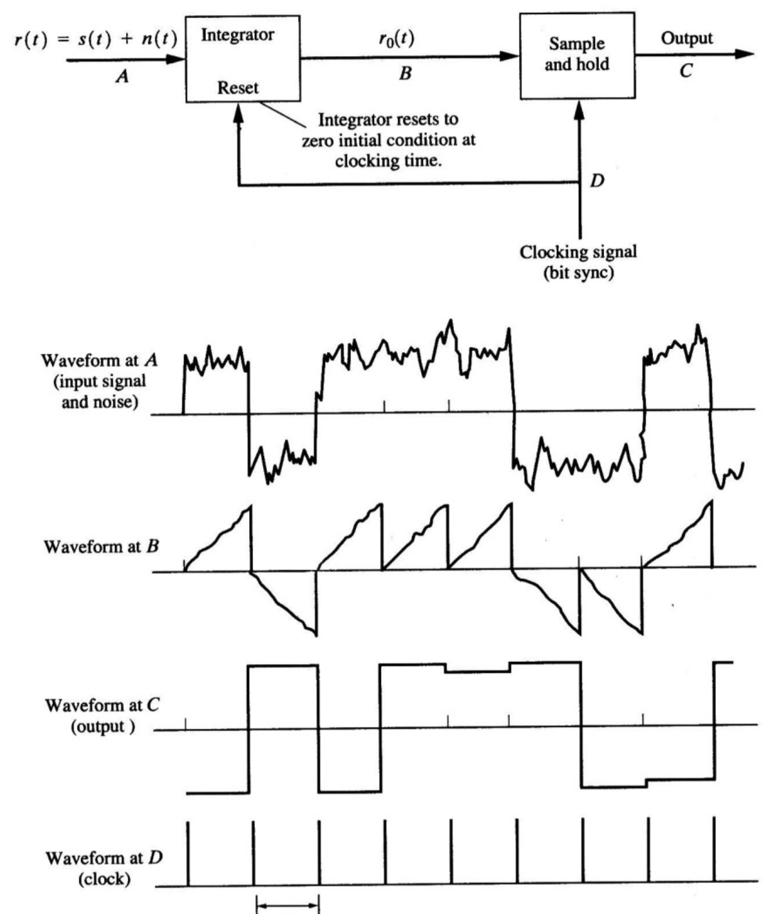


Figure 6–17 Integrate-and-dump realization of a matched filter.

Analysis of Signal and Noise

- The signal is referred to a deterministic waveform, modeled and analyzed by mathematical tool “Signals and Systems”.
- The noise is referred to a random waveform, modeled and analyzed by mathematical tool “Random Processes”

Review of “Signals and Systems”

- Fourier Transform and Spectra (LC 2-2)
- Power Spectral Density and Autocorrelation Function (LC 2-3)
- Orthogonal Series Representation (LC 2-4)
- Fourier Series (LC 2-5)
- Linear Systems (LC 2-6)
- Sampling Theorem (LC 2-7)
- Discrete Fourier Transform (LC 2-8)

Practical Waveforms

- Practical waveforms that are physically realizable (i.e., measurable in a laboratory) satisfy several conditions:
  - The waveform has significant (nonzero) values over a composite time interval that is finite.
  - The spectrum of the waveform has significant values over a composite frequency interval that is finite.
  - The waveform is a continuous function of time.
  - The waveform has a finite peak value.
  - The waveform has only real values. That is, at any time, it cannot have a complex value $a + jb$, where b is nonzero.

Mathematical Models

- Mathematical models that violate some or all of the conditions listed above are often used, and for one main reason – to simplify the mathematical analysis.
- The physical waveform is said to be an energy signal because it has finite energy, whereas the mathematical model is said to be a power signal because it has the property of finite power (and infinite energy).

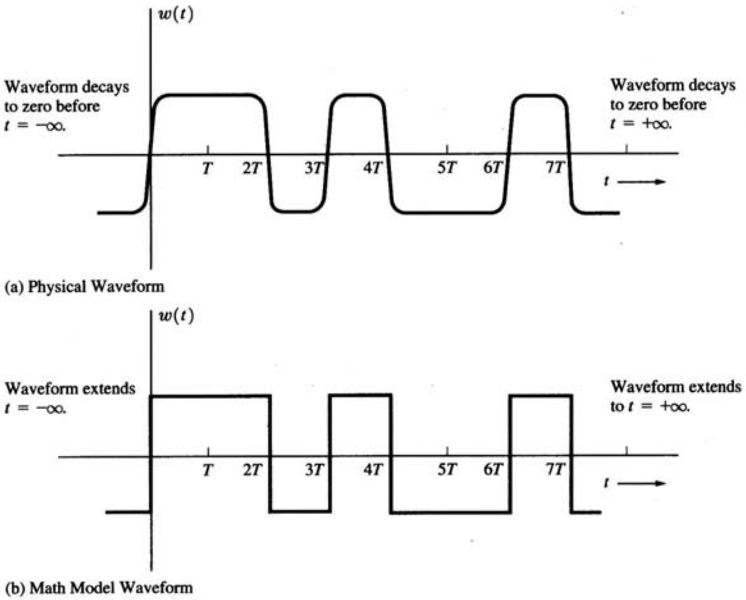


Figure 2–1 Physical and mathematical waveforms.

Time Average Operator

- The time average operator is given by

$$<[\cdot]>=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}[\cdot]dt$$

- Linear Operator, the average of the sum of two quantities is equal to the sum of their averages

$$\langle a_1 w_1(t)+a_2 w_2(t)\rangle = a_1\langle w_1(t)\rangle + a_2\langle w_2(t)\rangle$$

- A waveform w(t) is periodic with period T0 if

$$w(t)=w(t+T_0)\qquad \text{for all } t$$

where T0 is the smallest positive number that satisfies the relationship.

- **THEOREM**: If the waveform involved is periodic with period T0, the time average operator can be reduced to

$$<[\cdot]>=\frac{1}{T_0}\int_{-T0/2+a}^{T0/2+a}[\cdot]dt$$

DC Value

- The dc ("direct current") value of a waveform *w(t)* is given by its time average, *<w(t)>*.

$$W_{dc} = \langle w(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} w(t)\, dt$$

- For any physical waveform, we are actually interested in evaluating the dc value only over a finite interval of interest, say, from t1 to t2, so that the dc value is

$$W_{dc} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} w(t)\, dt$$

Power

- Let *v(t)* denote the voltage across a set of circuit terminals, and let i*(t)* denote the current into the terminal. The instantaneous power associated with the circuit is given by

$$p(t) = v(t)i(t)$$

- The average power is

$$P = < p(t) > = < v(t)i(t) >$$

- The root-mean-square (rms) value of *w(t)* is

$$W_{rms} = \sqrt{< w^2(t) >}$$

- If the load is resistive, the average power is

$$P = \frac{< v^2(t) >}{R} = < i^2(t) > R = \frac{V_{rms}^2}{R} = I_{rms}^2 R = V_{rms} I_{rms}$$

Power

- The concept of normalized power is often used by communication engineers. In this concept, *R* is assumed to be 1 *Ω*, although it may be another value in the actual circuit. The average normalized power is

$$P = \langle w^2(t) \rangle$$

- *w(t)* is a power waveform if and only if the normalized average power *P* is finite and nonzero. ($0 < P < \infty$)
- The total normalized energy is

$$E = \lim_{T \to \infty} \int_{-T/2}^{T/2} w^2(t)\,dt$$

- *w(t)* is an energy waveform if and only if the total normalized energy *E* is finite and nonzero. ($0 < E < \infty$)

Power

- Physically realizable waveforms are of the energy type.
- Basically, mathematical models are of the power type.
- Mathematical functions can be found that have both infinite energy and infinite power, e.g., $w(t) = e^{-t}$.

Decibel

- The decibel gain of a circuit is

$$dB = 10\log\left(\frac{P_{out}}{P_{in}}\right)$$

- The decibel signal-to-noise ratio is

$$(S/N)_{dB} = 10\log\frac{P_{signal}}{P_{noise}} = 10\log\frac{<s^2(t)>}{<n^2(t)>}$$

$$= 10\log\frac{V^2_{rms\ signal}}{V^2_{rms\ noise}} = 20\log\frac{V_{rms\ signal}}{V_{rms\ noise}}$$

- The decibel power level with respect to 1 milliwatt is

$$dBm = 10\log\frac{P(watts)}{10^{-3}(watts)} = 30 + 10\log\frac{P(watts)}{1(watts)} = 30 + dBW$$

Bandlimited Signal

- A waveform *w(t)* is said to be (absolutely) bandlimited to *B* hertz if

$$W(f) = 0, \quad \text{for} \left| f \right| \geq B$$

- A waveform *w(t)* is said to be (absolutely) time limited if

$$w(t) = 0, \quad \text{for} \left| t \right| > T$$

- An absolutely bandlimited waveform cannot be absolutely time limited, and vice versa. This raises an engineering paradox. This paradox is resolved by realizing that we are modeling a physical process with a mathematical model and perhaps the assumption in the model are not satisfied – although we believe them to be satisfied.
- If a waveform is absolutely bandlimited, it is analytic function.

On Bandwidth, D. Slepian, Proceedings of IEEE, vol. 64, no. 3, 1976

Bandwidth

- What is bandwidth? There are numerous definitions of the term.
- In engineering definitions, the bandwidth is taken to be the width of a positive frequency band. In other words, the bandwidth would be $f_2 - f_1$, where $f_2 \geq f_1 \geq 0$ and is determined by the particular definition that is used.
- ABSOLUTE BANDWIDTH is $f_2 - f_1$, where the spectrum is zero outside the interval $f_1 < f < f_2$ along the positive frequency axis.
- 3-dB BANDWIDTH (or HALF-POWER BANDWIDTH) is $f_2 - f_1$, where for frequencies inside the band $f_1 < f < f_2$, the magnitude spectra, say, $|H(f)|$, falls no lower than $1/\sqrt{2}$ times the maximum value of $|H(f)|$, and the maximum value occurs at a frequency inside the band.
- EQUIVALENT NOISE BANDWIDTH is the width of a fictitious rectangular spectrum such that the power in that rectangular band is equal to the power associated with the actual spectrum over positive frequencies. Let f_0 be the frequency at which the magnitude spectrum has a maximum, then

$$B_{eq}|H(f_0)|^2 = \int_0^\infty |H(f)|^2 df \Rightarrow B_{eq} = \frac{1}{|H(f_0)|^2}\int_0^\infty |H(f)|^2 df$$

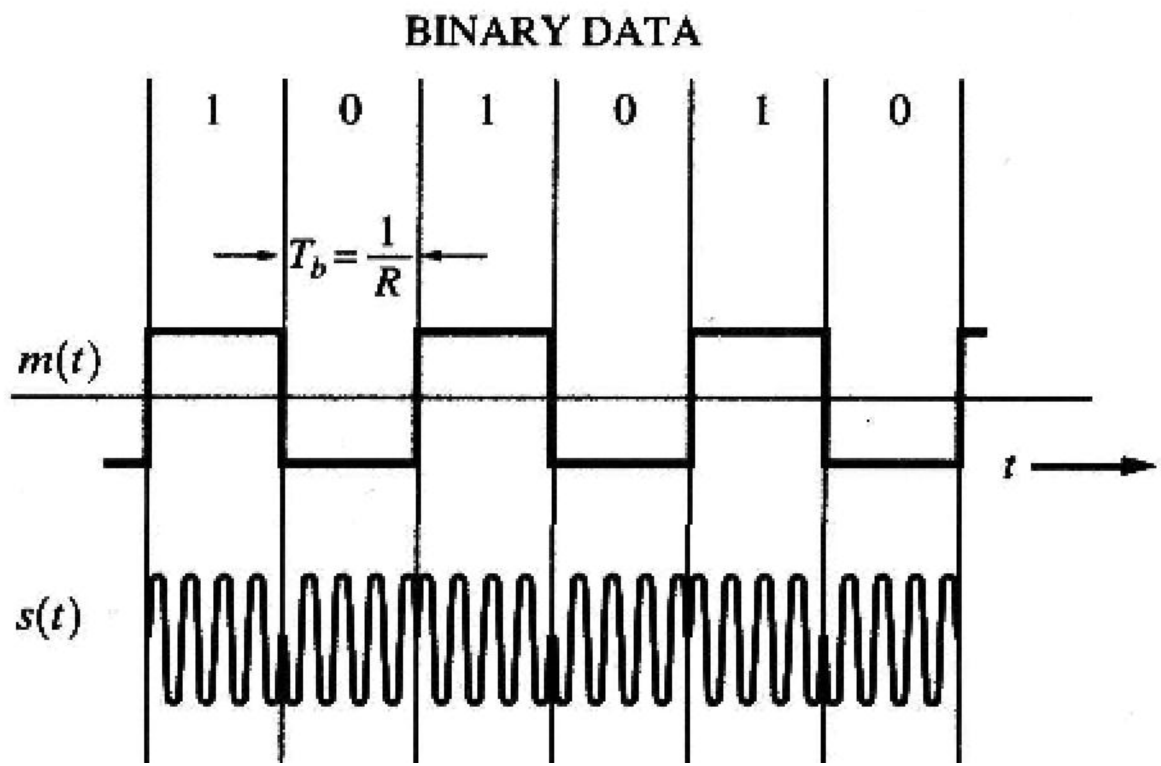
Bandwidth

- NULL-TO-NULL BANDWIDTH (or ZERO-CROSSING BANDWIDTH) is $f_2 - f_1$, where f_2 is the first null in the envelope of the magnitude spectrum above f_0 and , for bandpass systems, f_1 is the first null in the envelope below f_0, where f_0 is the frequency where the magnitude spectrum is a maximum. For baseband systems f_1 is usually zero.
- BOUNDED SPECTRUM BANDWIDTH is $f_2 - f_1$ such that outside the band $f_1 < f < f_2$, the PSD must be down at least a certain amount, say, 50dB, below the maximum value of the PSD.
- POWER BANDWIDTH is $f_2 - f_1$, where $f_1 < f < f_2$ defines the frequency band in which 99% of the total power resides.
- FCC BANWIDTH is an authorized bandwidth parameter assigned by the FCC to specify the spectrum allowed in communication systems. For operating frequencies below 15 GHz, in any 4kHz band, the center frequency of which is removed from the assigned frequency by P% (50 < P < 250) of the authorized bandwidth (B, in MHz), the attenuation below the mean output power level is given by the following equation:

$$A = 35 + 0.8(P - 50) + 10 \log B \quad \text{(dB)}$$

Example

- BANDWIDTH FOR A BPSK SIGNAL $s(t) = m(t)\cos\omega_c t$



Example

- The worst-case (widest-bandwidth) spectrum occurs when the digital modulating waveform has transitions that occur most often. In this case, m(t) would be a square wave.

$$P_m(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0)$$

$$c_n = \begin{cases} \dfrac{1}{T_0}\displaystyle\int_{-T_0/2}^{T_0/2} m(t)dt = 0 \\ f_0 \cdot 2 \cdot T_b Sa(\pi n f_0 T_b) = Sa(\pi n / 2) \end{cases}$$

$$P_m(f) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} Sa^2(\pi n/2)\delta(f - nf_0) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} Sa^2(\pi n/2)\delta(f - n\frac{R}{2})$$

Example

- The autocorrelation of s(t).

$$R_s(\tau) = < s(t)s(t+\tau) > = < m(t)\cos\omega_c t \cdot m(t+\tau)\cos\omega_c(t+\tau) >$$

$$= < m(t)m(t+\tau)\cdot\tfrac{1}{2}[\cos\omega_c\tau + \cos(2\omega_c t + \omega_c\tau)] >$$

$$= \tfrac{1}{2} < m(t)m(t+\tau)\cos\omega_c\tau > + \tfrac{1}{2} < m(t)m(t+\tau)\cos(2\omega_c t + \omega_c\tau) >$$

$$= \tfrac{1}{2} < m(t)m(t+\tau) > \cos\omega_c\tau$$

$$= \tfrac{1}{2} R_m(\tau)\cos\omega_c\tau$$

- The PSD of s(t) is obtained by taking Fourier transform of both sides

$$P_s(f) = F[R_s(\tau)] = F[\tfrac{1}{2} R_m(\tau)\cos\omega_c\tau] = \tfrac{1}{4}[P_m(f-f_c) + P_m(f+f_c)]$$

$$= \tfrac{1}{4}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} Sa^2(\pi n/2)[\delta(f-f_c-n\frac{R}{2}) + \delta(f+f_c-n\frac{R}{2})]$$

Example

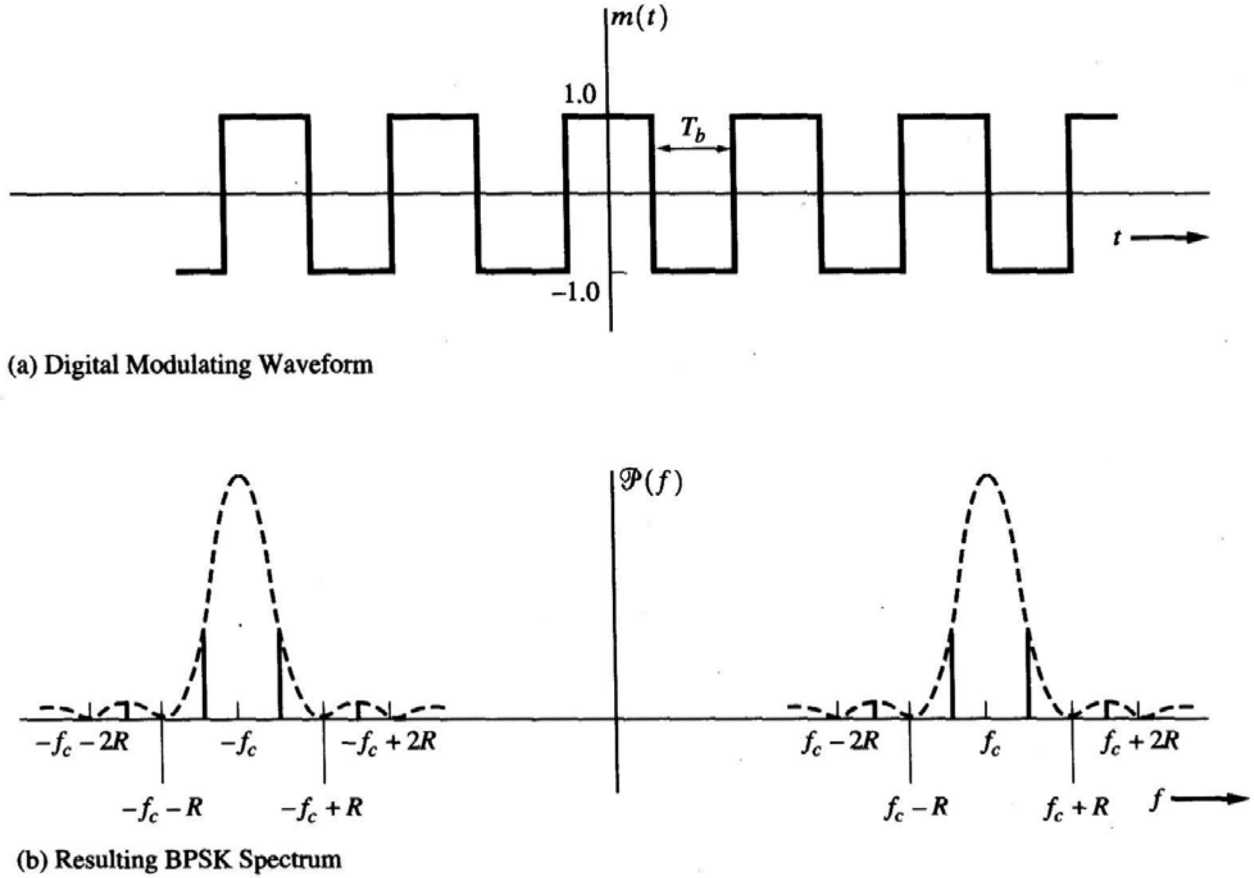
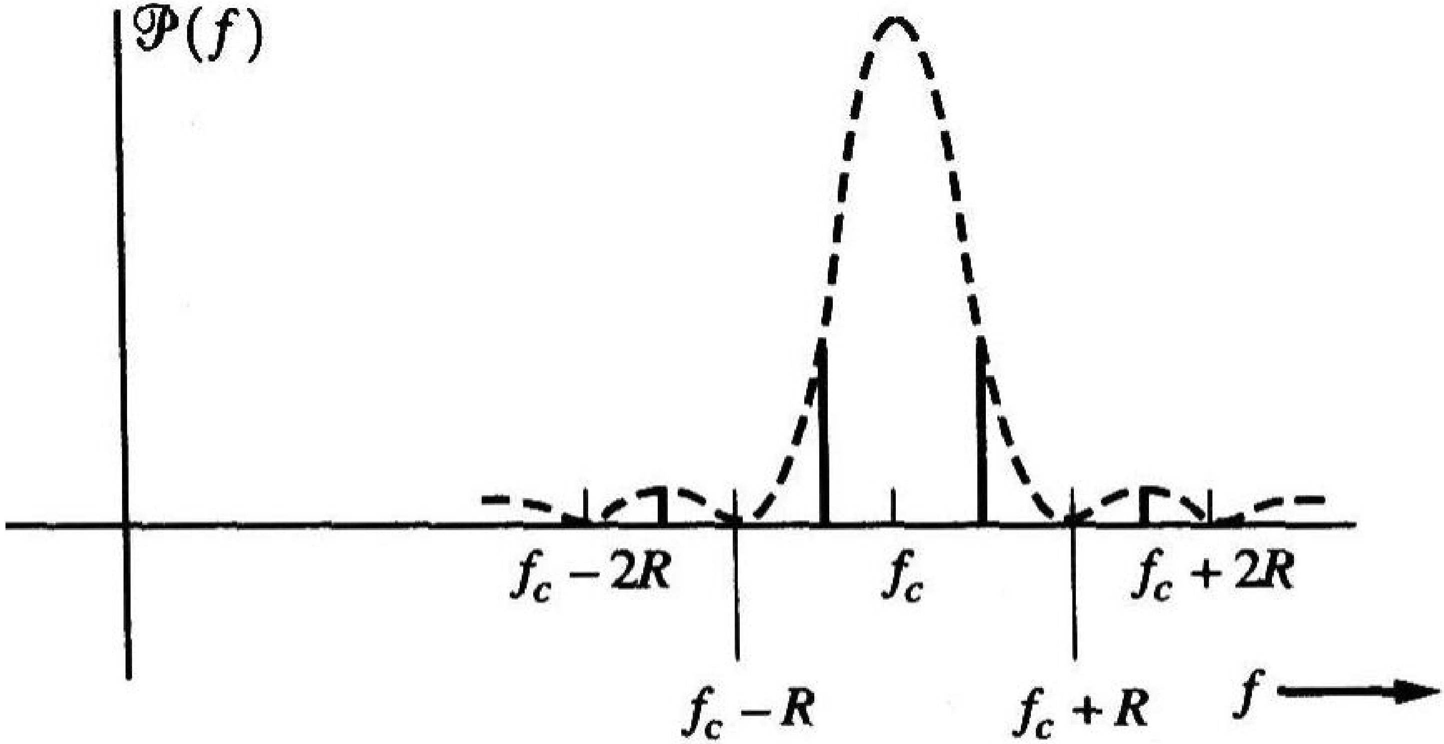


Figure 2–23 Spectrum of a BPSK signal.

Example

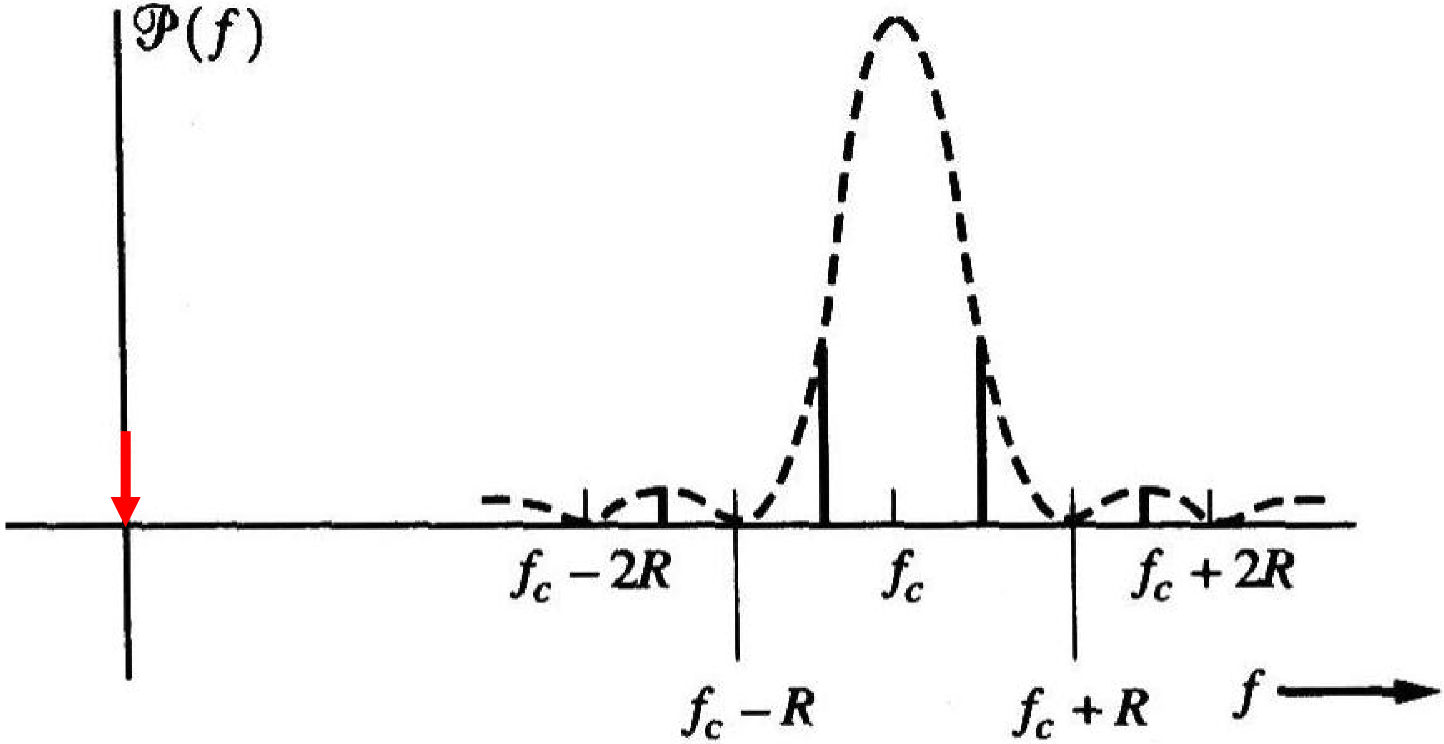
- To evaluate the bandwidth for the BPSK signal, the shape of the PSD for the positive frequencies is needed, it is

$$P_s(f) = \frac{1}{4}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} Sa^2[\pi T_b(f-f_c)]\delta(f-f_c-n\frac{R}{2})$$



Absolute Bandwidth

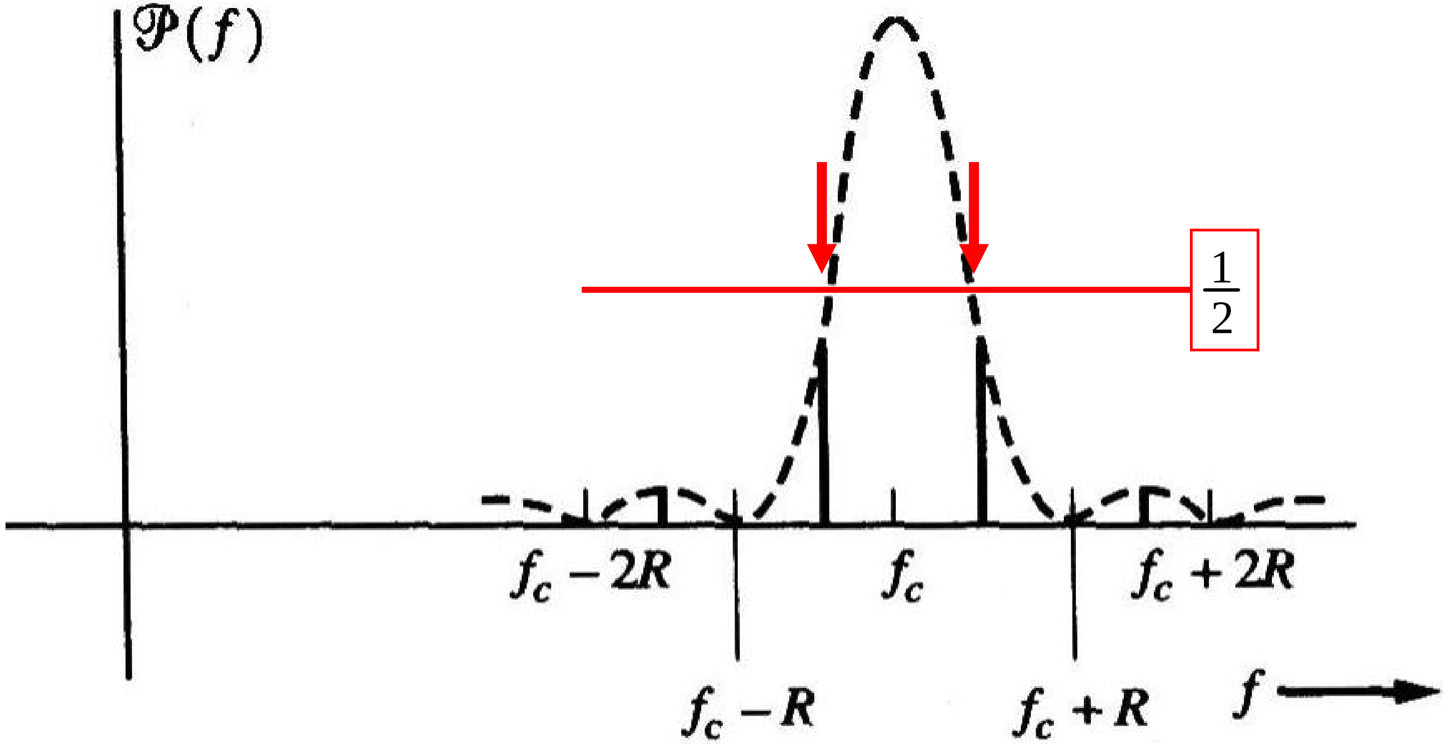
$$B_{abs} = f_2 - f_1 = \infty - 0 = \infty$$



3-dB Bandwidth

$$Sa^2[\pi T_b(f_2 - f_c)] = \tfrac{1}{2} \Rightarrow \pi T_b(f_2 - f_c) \approx 1.4 \Rightarrow$$

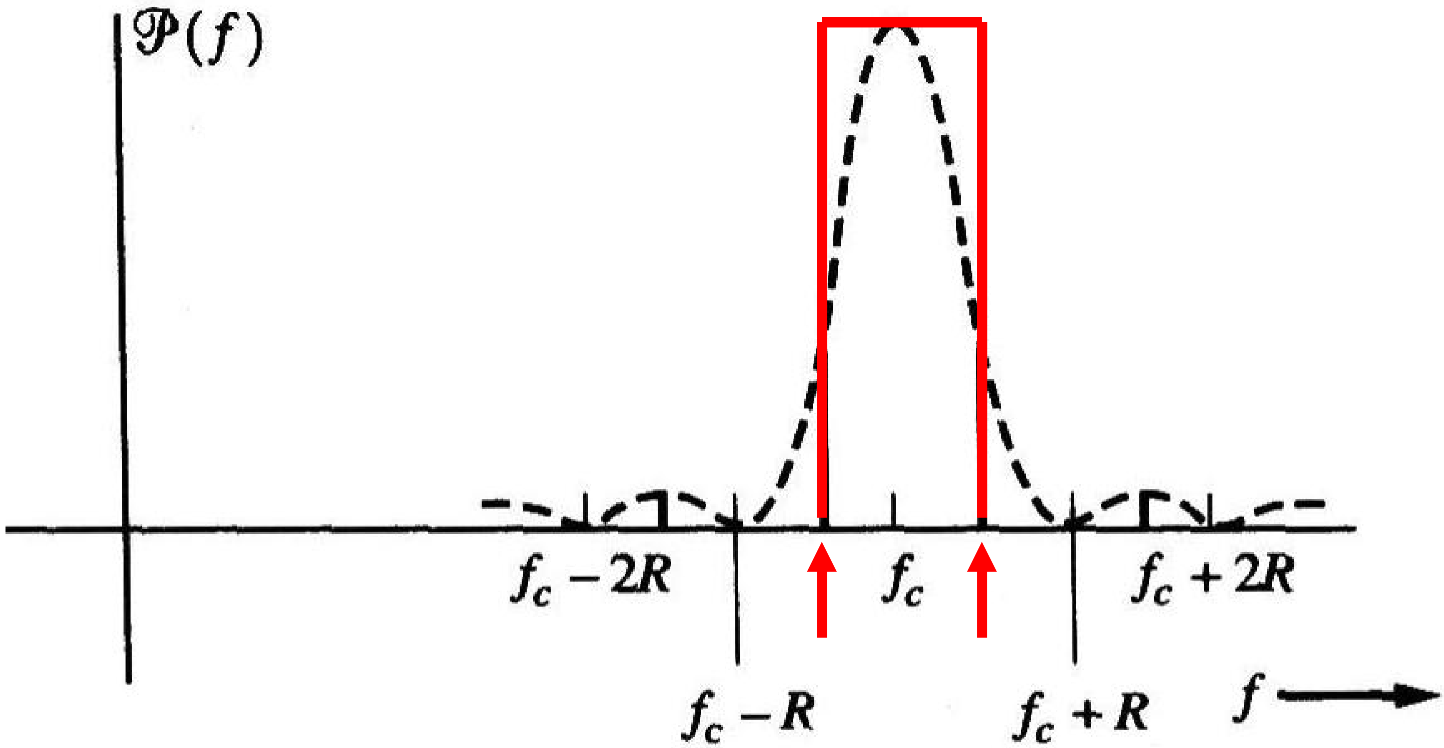
$$f_2 - f_c = \frac{1.4}{\pi T_b} \approx \frac{0.446}{T_b} = 0.446R \Rightarrow B_{3dB} = f_2 - f_1 = 2(f_2 - f_c) = 0.891R$$



Equivalent Noise Bandwidth

$$\int_0^\infty Sa^2[\pi T_b(f-f_c)]df \approx 2\int_{f_c}^\infty Sa^2[\pi T_b(f-f_c)]df = 2\frac{R}{\pi}\int_0^\infty Sa^2 x dx = 2\frac{R}{\pi}\cdot\frac{\pi}{2} = R$$

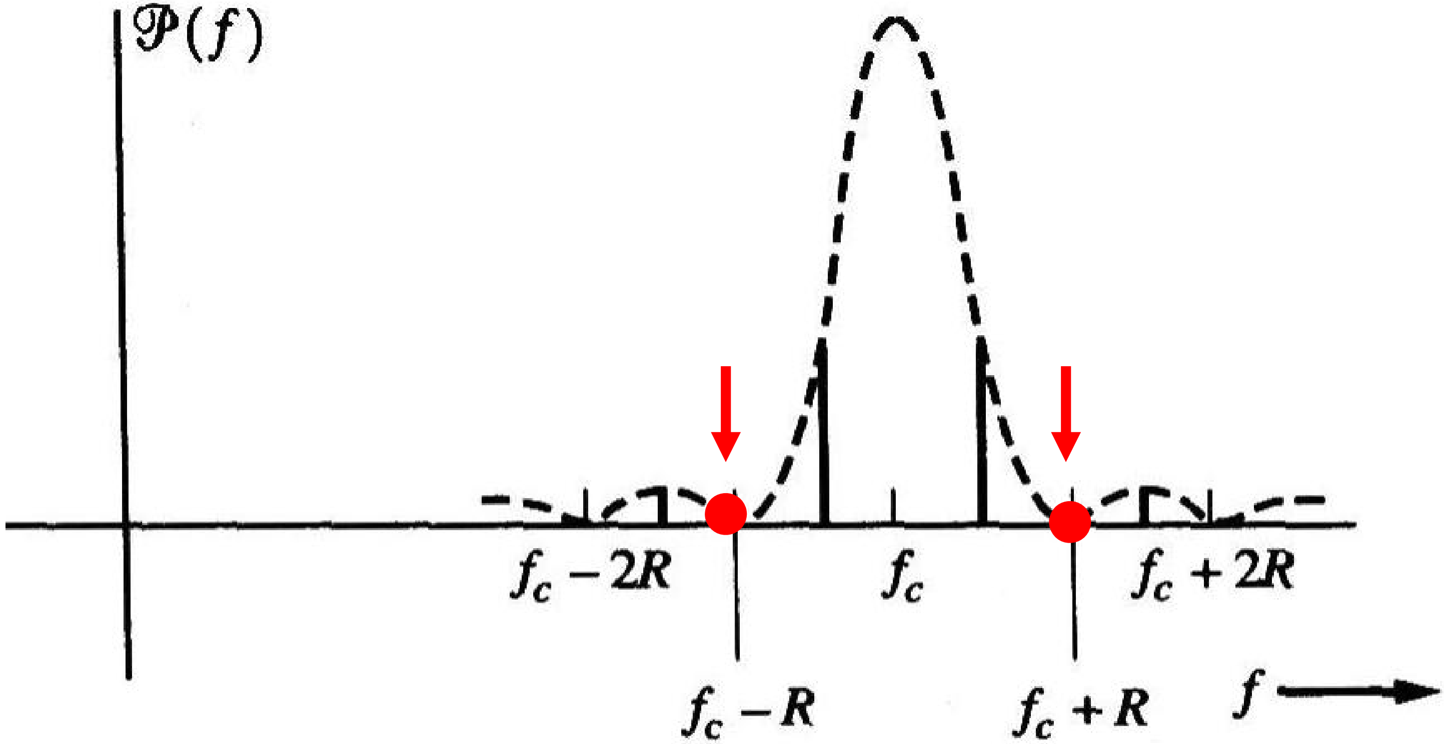
$$Sa^2[\pi T_b(f-f_c)]\Big|_{f=f_c} = 1 \Rightarrow B_{eq} = \frac{1}{|H(f_0)|^2}\int_0^\infty |H(f)|^2 df = \frac{R}{1} = 1.00R$$



Null-to-Null Bandwidth

$$Sa^2[\pi T_b(f - f_c)] = 0 \Rightarrow \pi T_b(f - f_c) = \pm\pi$$

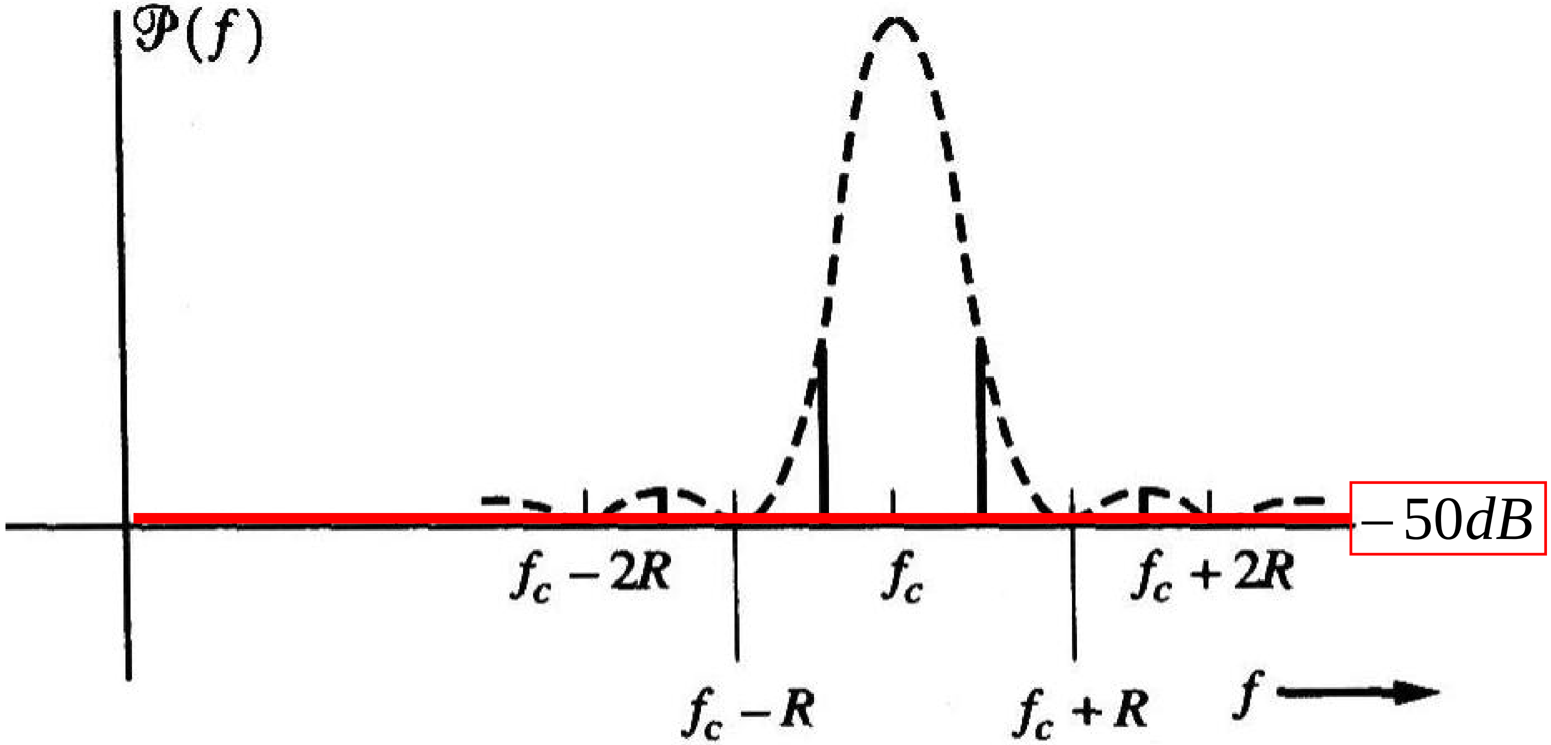
$$\Rightarrow f_2 = f_c + \frac{1}{T_b} = f_c + R, \quad f_1 = f_c - \frac{1}{T_b} = f_c - R \Rightarrow B_{null} = 2R = 2.00R$$



Bounded spectrum bandwidth (50dB)

$$10\log Sa^2[\pi T_b(f-f_c)] \le 10\log\frac{1}{[\pi T_b(f-f_c)]^2} = -10\log[\pi T_b(f-f_c)]^2 \le -50$$

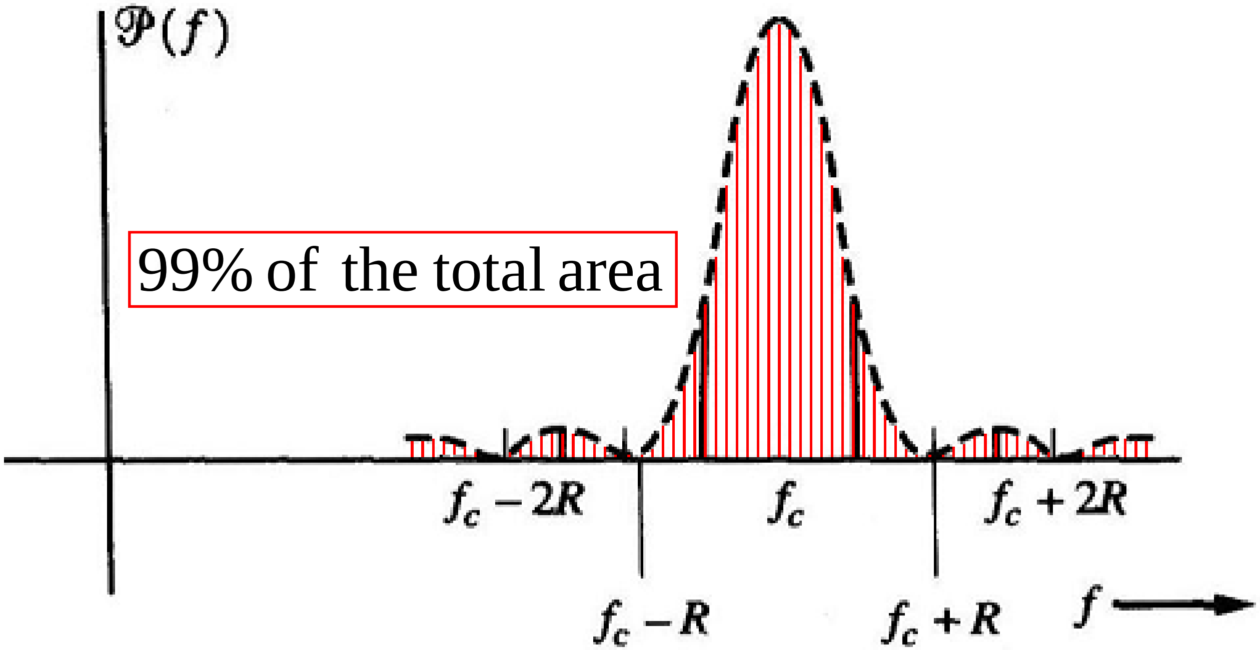
$$[\pi T_b(f-f_c)]^2 \ge 10^{\frac{50}{10}} = 10^5 \Rightarrow f-f_c \ge \frac{\sqrt{10^5}}{\pi T_b} \approx 100.6584R \Rightarrow B_{50dB} = 2\times 100.6584R \approx 201.32R$$



Power bandwidth (20dB)

Let $x = \pi T_b f$

$$\frac{\int_0^{x_0} Sa^2(x)dx}{\int_0^{\infty} Sa^2(x)dx} = 0.99 \Rightarrow x_0 \approx 32.29 \Rightarrow f_0 = \frac{1}{\pi T_b} x_0 \approx 10.278R \Rightarrow B_{power} = 2 f_0 \approx 20.56R$$



FCC bandwidth

- In this example, the FCC bandwidth parameter B = 30 MHz. Assume that the PSD is constant across the 4-kHz bandwidth (4 kHz / 30 MHz = 0.013%), the decibel attenuation of the BPSK signal

$$A(f) = -10\log\frac{\mathrm{P}_{4kHz}(f)}{\mathrm{P}_{total}} = -10\log\frac{4000P(f)}{B_{eq}P(f_c)}$$

Plugging in B_{eq} = R Hz, $P(f_c)$ = 1 Watt/Hz, we have

$$A(f) = -10\log\frac{4000}{R}Sa^2[\pi T_b(f - f_c)]$$

The envelope of A(f)

$$A_{env}(f) = -10\log\frac{4000}{R}\frac{1}{[\pi T_b(f - f_c)]^2} = -10\log\frac{4000R}{[\pi(f - f_c)]^2}$$

$$= -10\log\frac{4000R}{[\pi(P\%B)]^2} = -10\log\frac{4000R}{[\pi(P\%\times 30\times 10^6)]^2} \approx 83.46 - 10\log\frac{R}{P^2}$$

FCC Bandwidth

- FCC envelope *A* for *B* = 30 MHz

$$A = 35 + 0.8(P - 50) + 10\log B$$
$$= 35 + 0.8(P - 50) + 10\log 30 \approx 49.77 + 0.8(P - 50)$$

$$A = 80dB \Rightarrow P = 50 + \frac{80 - 49.77}{0.8} \approx 87.79$$
$$P = 87.79 \Rightarrow A_{env}(f) = 83.46 - 10\log\frac{R}{87.79^2}$$
$$\approx 122.33 - 10\log R \geq 80 \Rightarrow R \leq 10^{\frac{122.33-80}{10}} \approx 17100\, bps \approx 0.0171\, Mbps$$

FCC Bandwidth

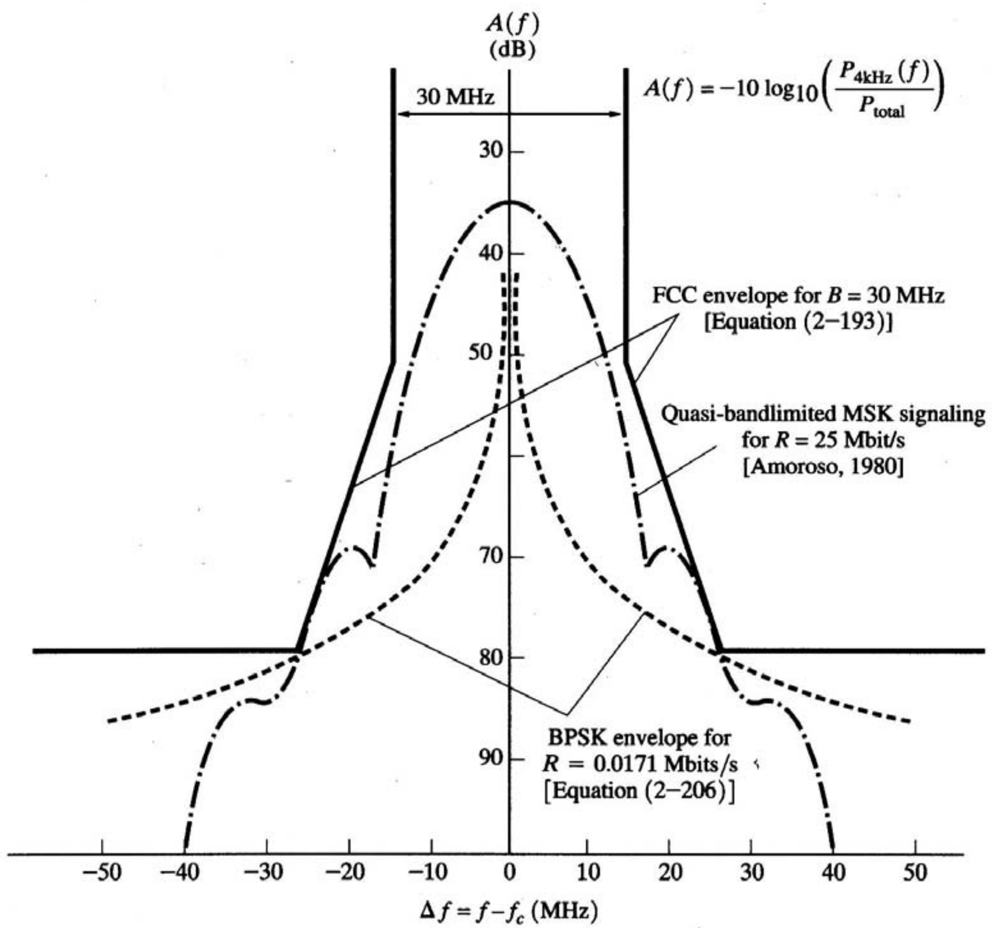


Figure 2–24 FCC-allowed envelope for $B = 30$ MHz.

Homework

- 2-4, 2-10, 2-32, 2-45, 2-92