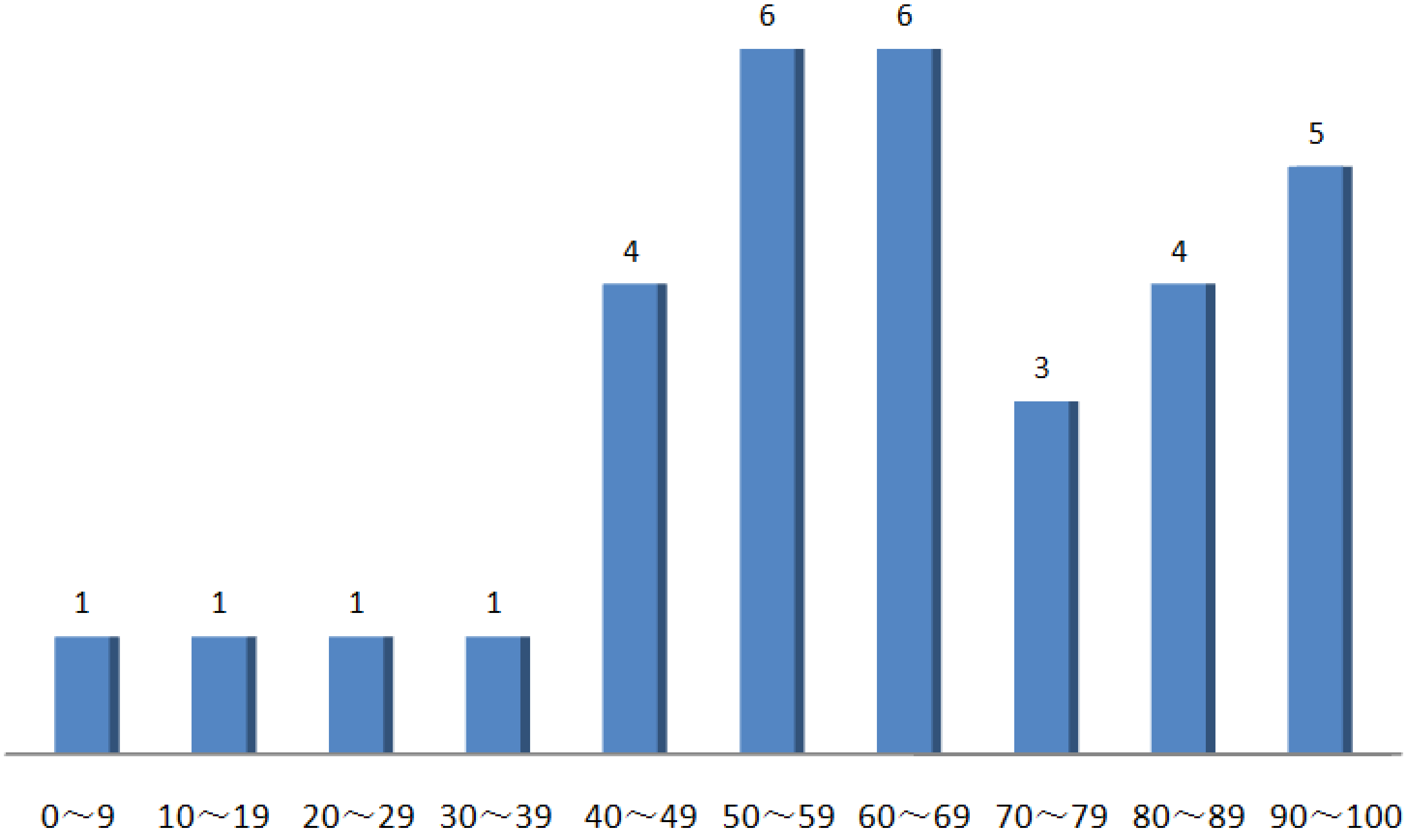
Distribution of Mid-Term



Failed Ratio = 14/32 = 43.75%

Baseband Digital System (6)

LC 6-8

Lecture 18, 2008-11-18

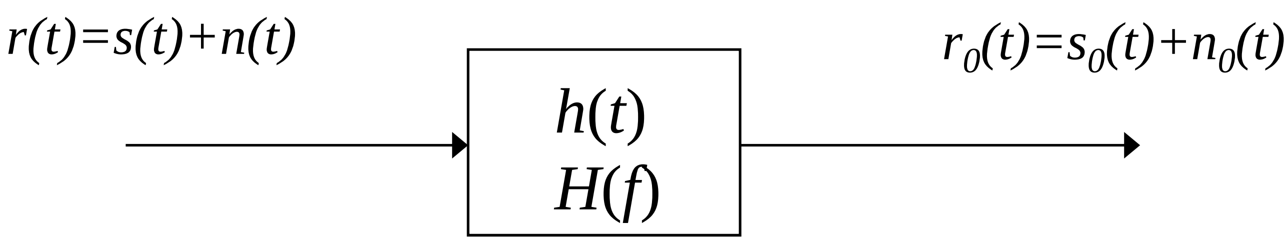
Contents

Matched Filter

- The matched filter is the linear filter that maximizes SNR and that has a transfer function given by

$$H(f) = K\frac{S^*(f)}{P_n(f)}e^{-j2\pi f t_0}$$

where S(f) = F[s(t)] is the Fourier transform of the know input signal s(t) of duration T sec. $P_n(f)$ is the PSD of the input noise, t_0 is the sampling time when $(S/N)_{out}$ is evaluated and K is an arbitrary real nonzero constant.



Matched Filter

Proof

The output signal at time t_0 is

$$s_0(t_0) = \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi f t_0} df$$

The average power of the output noise is

$$\overline{n_0^2(t)} = R_{n0}(0) = \int_{-\infty}^{\infty} |H(f)|^2 P_n(f) df$$

The signal to noise ratio is

$$\left(\frac{S}{N}\right)_{out} = \frac{\left| \int_{-\infty}^{\infty} H(f)S(f)e^{j2\pi f t_0} df \right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 P_n(f) df}$$

Proof

Schwarz inequality

$$\left| \int_{-\infty}^{\infty} A(f)B(f)df \right|^2 \le \int_{-\infty}^{\infty} |A(f)|^2 df \int_{-\infty}^{\infty} |B(f)|^2 df$$

The equality is obtained only when $A(f) = KB^*(f)$

Let $A(f) = H(f)\sqrt{P_n(f)}, \quad B(f) = \dfrac{S(f)e^{j2\pi ft_0}}{\sqrt{P_n(f)}}$

$$\left(\frac{S}{N}\right)_{out} \le \frac{\int_{-\infty}^{\infty} |H(f)|^2 P_n(f)df \int_{-\infty}^{\infty} \frac{|S(f)|^2}{P_n(f)} df}{\int_{-\infty}^{\infty} |H(f)|^2 P_n(f)df} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{P_n(f)} df$$

When $H(f)\sqrt{P_n(f)} = \dfrac{KS^*(f)e^{-j2\pi ft_0}}{\sqrt{P_n(f)}}$, namely $H(f) = \dfrac{KS^*(f)e^{-j2\pi ft_0}}{P_n(f)}$

The maximum $(S/N)_{out}$ is obtained.

Results for White Noise

- For the case of white noise, $P_n(f) = N_0 / 2$

$$H(f) = \frac{2KS^*(f)e^{-j2\pi f t_0}}{N_0}$$

- Theorem. When the input noise is white, the impulse response of matched filter becomes

$$h(t) = Cs(t_0 - t)$$

Where C is an arbitrary real positive constant, t0 is the time of the peak signal output, and s(t) is the known input-signal waveshape.

Proof

$$h(t) = F^{-1}\left[H(f)\right] = \frac{2K}{N_0}\int_{-\infty}^{\infty} S^*(f)e^{-j2\pi f t_0}e^{j2\pi f t}df$$

$$= \frac{2K}{N_0}\left[\int_{-\infty}^{\infty} S(f)e^{j2\pi f(t_0-t)}df\right]^* = \frac{2K}{N_0}\left[s(t_0-t)\right]^*$$

- The output SNR

$$\left(\frac{S}{N}\right)_{out} = \int_{-\infty}^{\infty}\frac{|S(f)|^2}{N_0/2}df = \frac{2}{N_0}\int_{-\infty}^{\infty}|s(t)|^2\,dt = \frac{2E_s}{N_0}$$

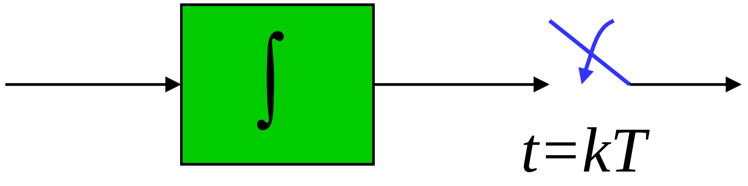
depends on the signal energy and PSD level of noise, and not on the particular signal wave shape that is used.

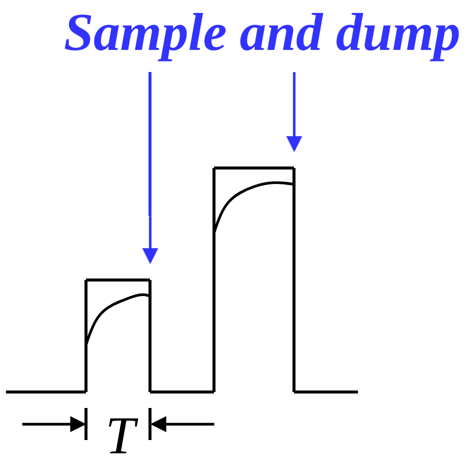
Example

- Rectangular pulse, 6-11

Integrate-and-Dump Filter

- First, integrate for *T* sec
- Second, sample at symbol period *T* sec
- Third, reset integration for next time period





- The output of the integrator:

$$V = \int_{t_0}^{t_0+T} [s(t) + n(t)]dt$$

$$= \begin{cases} AT + N & A \quad is \quad sent \\ -AT + N & -A \quad is \quad sent \end{cases}$$

$$N = \int_{t_0}^{t_0+T} n(t)\, dt$$ is a random variable.

- N is Gaussian. Why?

$$E[N] = E[\int_{t_0}^{t_0+T} n(t)dt] = \int_{t_0}^{t_0+T} E[n(t)]dt = 0$$

$$Var[N] = E[N^2] - E^2[N]$$

$$= E[N^2]$$

$$= E\left\{\left[\int_{t_0}^{t_0+T} n(t)dt\right]^2\right\}$$

$$= \int_{t_0}^{t_0+T}\int_{t_0}^{t_0+T} E[n(t)n(s)]dtds$$

$$= \int_{t_0}^{t_0+T}\int_{t_0}^{t_0+T} \frac{N_0}{2}\delta(t-s)dtds$$

$$= \frac{N_0 T}{2}$$

White noise is uncorrelated

Therefore, the pdf of N is:

$$f_N(n) = \frac{e^{-n^2/(N_0 T)}}{\sqrt{\pi N_0 T}}$$

Correlation Processing

- **Theorem**, For the case of white noise, the matched filter may be realized by correlating the input with s(t), that is

$$r_0(t_0) = \int_{t_0-T}^{t_0} r(t)s(t)dt$$

where s(t) is the known signal waveshape and r(t) is the processor input.

- **Proof**

The output of the matched filter at time t_0 is

$$r_0(t_0) = r(t_0) * h(t_0) = \int_{-\infty}^{t_0} r(\lambda)h(t_0-\lambda)d\lambda$$

$$\because \quad h(t) = \begin{cases} s(t-t_0), & 0 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

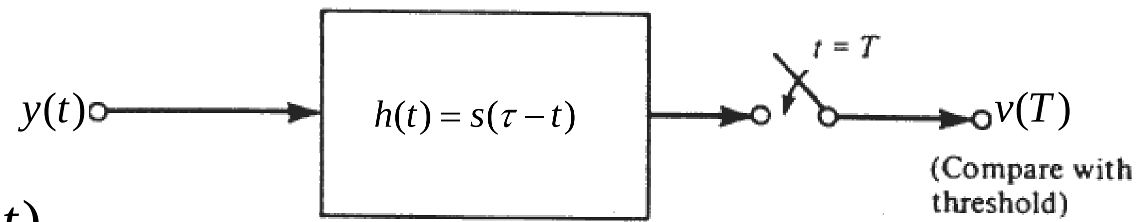
$$\therefore \quad r_0(t_0) = \int_{t_0-T}^{t_0} r(t)s[t_0-(t_0-t)]dt = \int_{t_0-T}^{t_0} r(t)s(t)dt$$

Example

- 6-12

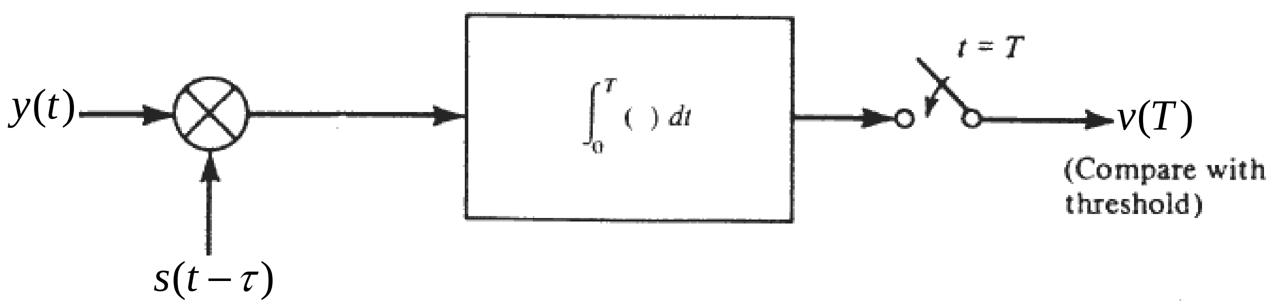
Convolution vs. Correlation

- The mathematical operation of matched filter (MF) is convolution.
- The mathematical operation of a correlator is correlation.
- The term “matched filter” is often used synonymously with “correlator”.



(a) Matched filter implementation

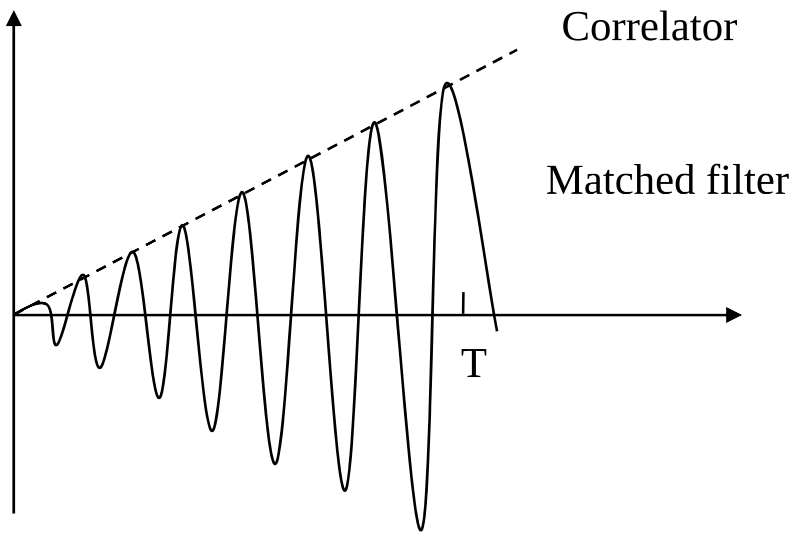
$$v(t) = s(\tau - t) \otimes y(t)$$
$$= \int_0^{\tau} s(t-\tau) y(\tau) d\tau$$



(b) Correlator implementation

$$v(t) = \int_0^{\tau} s(t-\tau) y(\tau) d\tau$$

- The process of convolving two signals reverses one of them in time.
- Convolution in the MF with a time-reversed function results in a second time-reversal.
- The output of correlator and matched filter for a sinusoidal wave



Homework

- LC 6-50, 6-51