Baseband Digital System (3)

LC 3-5

Lecture 14, 2008-10-31

Contents

- Digital signal representation
- Binary and multilevel signals
- Line codes and their Spectra
- Spectral efficiency

Digital Signaling

- Mathematical Representation of the waveform
- Estimation of Bandwidth
- Voltage (or current) waveform for digital signals

$$w(t) = \sum_{k=1}^{N} w_k \varphi_k(t) \qquad 0 < t < T_0$$

$w(t)$: PCM word or message

w_k : digital data

$\varphi_k(t)$: orthogonal functions

N : number of dimensions

T_0 : message time span

- Example: Message 'X' from a digital source - code word "0001101"

$$w_k : \ w_1 = 0 \ w_2 = 0 \ w_3 = 0 \ w_4 = 1 \ w_5 = 1 \ w_6 = 0 \ w_7 = 1$$

$$\{w_1, w_2, w_3, w_4, w_5, w_6, w_7\} = \{0001101\}$$

$$N = 7$$

- Baud (symbol rate)

$$D = N/\ T_0 \text{ symbols/sec}$$

$$N : \text{number of symbols (dimensions) used in } T_0 \text{ sec}$$

- Bit rate (information rate)

$$R = n/\ T_0 \text{ bits/sec}$$

$$N : \text{number of bits sent in } T_0 \text{ sec}$$

- Binary signal vs. multilevel signal

$$w_k \begin{cases} \text{binary (two) values - bianry signal} \\ \text{more than two values - multilevel signal} \end{cases}$$

- How to detect the data at the receiver? (Matched filter detection)

$$w_k = \frac{1}{K_k}\int_0^{T_0} w(t)\varphi_k^*(t)dt;\ k = 1,\ 2,\ \ldots,\ N$$

$$w(t)\text{: waveform at the receiver input}$$

$$\varphi_k(t)\text{: orthogonal functions}$$

Orthogonal Vector Space

$$\mathbf{w} = \sum_{j=1}^{N} w_j \varphi_j \quad \text{or}$$

$$\mathbf{w} = \left(w_1, w_2, w_3, \ldots, w_N \right)$$

$$\mathbf{w} : N\text{-dimensional vector}$$

$$N : \text{dimensional vector}$$

$$\left\{ \varphi_j \right\} : \text{orthogonal functions and would be called orthonormal functions if}$$

$$K_j = \int_0^{T_0} \varphi_j(t) \varphi_j^*(t) dt = \int_0^{T_0} \left| \varphi_j(t) \right|^2 dt = 1$$

Example

- This 3-bit (binary) signal could be directly represented by

$$s(t)=\sum_{j=1}^{3} d_j\, p\left[t-\left(j-\frac{1}{2}\right)T\right]=\sum_{j=1}^{3} d_j\, p_j(t)$$

$$p_j(t): \text{orthogonal functions}$$

$$p_j(t)=\begin{cases}5, & (j-1)T<t<jT\\ 0, & t \text{ otherwise}\end{cases}$$

$$\mathbf{d}=(d_1, d_2, d_3)=(1,0,1)$$

- The orthonormal series representation

$$s(t)=\sum_{j=1}^{M=3} s_j \varphi_j(t)$$

$$\varphi_j(t): \text{orthonormal functions}$$

$$\int_0^{T_0} \varphi_j(t)\varphi_j^*(t)\,dt=1$$

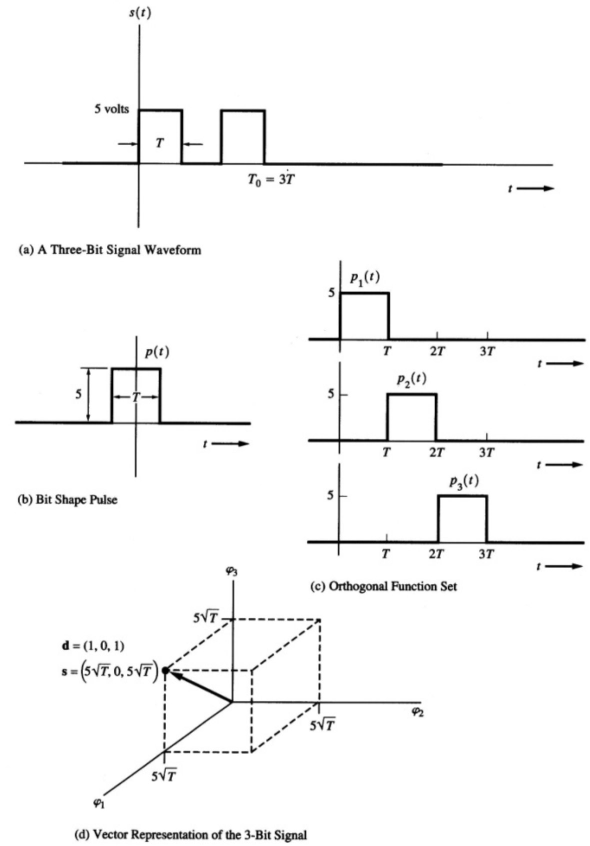


Figure 3–11 Representation for a 3-bit binary digital signal.

Example (con't)

Let $p_j(t) = A\varphi_j(t)$

$$K_j = \int_0^{T_0} p_j(t)\, p_j^*(t)dt = A^2 \int_0^{T_0} \varphi_j(t)\varphi_j^*(t)dt = A^2$$

$$A = \sqrt{K_j}$$

$$\varphi_j(t) = \frac{p_j(t)}{\sqrt{K_j}} = \frac{p_j(t)}{\sqrt{\int_0^{T_0} p_j(t)\, p_j^*(t)dt}} = \frac{p_j(t)}{\sqrt{\int_0^{T_0} p_j^2(t)dt}} = \frac{p_j(t)}{5\sqrt{T}} = \begin{cases} \frac{1}{\sqrt{T}}, & (j-1)T < t < jT \\ 0, & t \text{ otherwise} \end{cases}$$

$$s(t) = \sum_{j=1}^{3} s_j \varphi_j(t) = \sum_{j=1}^{3} d_j p_j(t)$$

$$s_j = d_j \frac{p_j(t)}{\varphi_j(t)} = d_j \sqrt{K_j} = d_j\left(5\sqrt{T}\right)$$

$$\mathbf{s} = \mathbf{d}\left(5\sqrt{T}\right) = (1, 0, 1)\left(5\sqrt{T}\right) = \left(5\sqrt{T}, 0, 5\sqrt{T}\right)$$

Bandwidth Estimation

- The bandwidth of the waveform $w(t)$ is

$$B \geq \frac{N}{2T_0} = \frac{1}{2}D$$

If the $\varphi_k(t)$ are the sinc type, the lower bound will be achieved.

Example

The number of messages $M = 256$

The number of bits $n = \log_2 M = 8$ bits

The time span $T_0 = 8$ ms

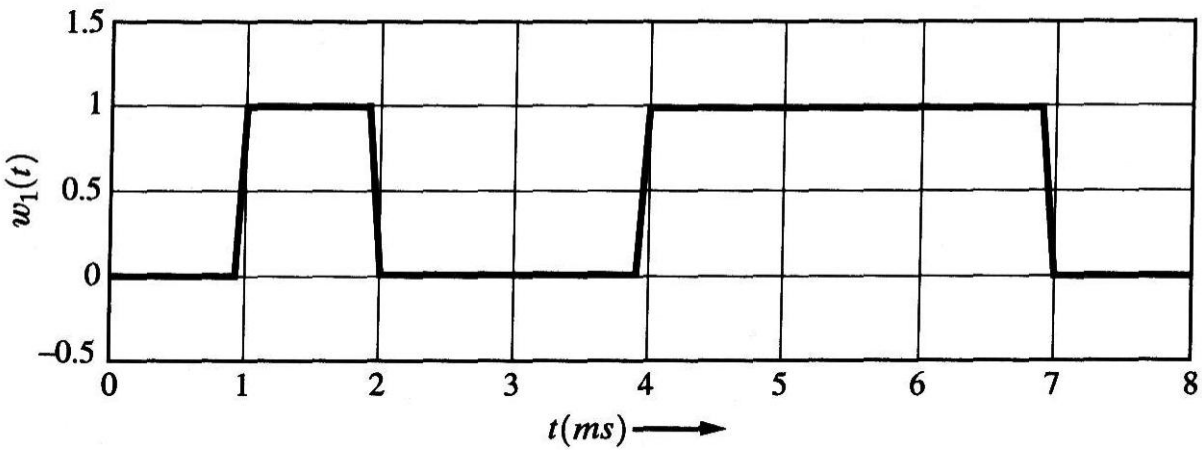
$$w(t) = \sum_{k=1}^{N} w_k \varphi_k(t) \qquad 0 < t < T_0$$

N : number of symbols, $N = n$ for binary signaling

A particular message corresponding to the code word 01001110

$w_1 = 0,\ w_2 = 1,\ w_3 = 0,\ w_4 = 0,\ w_5 = 1,\ w_6 = 1,\ w_7 = 1,\ w_8 = 0$

- CASE 1. RECTANGULAR PULSE ORTHOGONAL FUNCTIONS



(a) Rectangular Pulse Shape, $T_b = 1\ ms$

$$\varphi_k(t) = \Pi\left(\frac{t-(k-\frac{1}{2})T_s}{T_s}\right)$$

$$T_b = \frac{T_0}{n} = \frac{8}{8} = 1\ \text{ms}$$

$$R = \frac{1}{T_b} = \frac{n}{T_0} = 1\ \text{kbits/s}$$

$$D = \frac{1}{T_s} = \frac{N}{T_0} = \frac{n}{T_0} = \frac{1}{T_b} = R = 1\ \text{kbaud}$$

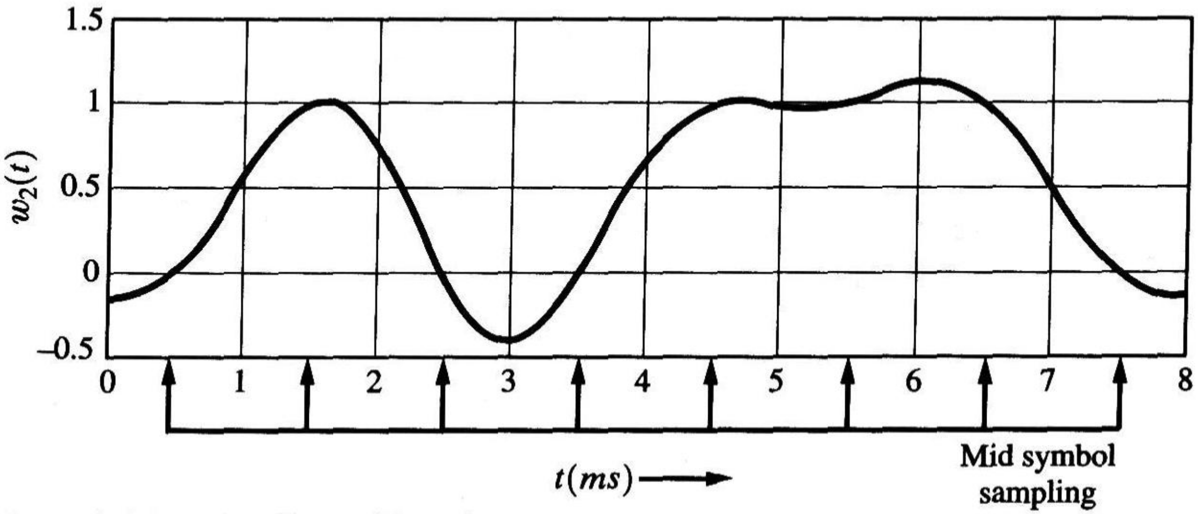
$$B_{null} = \frac{1}{T_s} = D = 1\ \text{kHz}$$

$$K_k = \int_0^{T_0} \varphi_k(t)\varphi_k^*(t)dt = \int_{(k-1)T_s}^{kT_s} (1\times 1)dt = T_s$$

$$w_k = \frac{1}{K_k}\int_0^{T_0} w(t)\varphi_k^*(t)dt = \frac{1}{T_s}\int_{(k-1)T_s}^{kT_s} w(t)dt = w(\tau)$$

$$\tau : \text{sampling time},\ (k-1)T_s < \tau < kT_s$$

- CASE 2. SINC PULSE ORTHOGONAL FUNCTIONS



(b) $\sin(x)/x$ Pulse Shape, $T_b = 1\ ms$

Figure 3–12 Binary signaling (computed).

$$\varphi_k(t) = Sa\left(\frac{\pi(t-(k-\frac{1}{2})T_s)}{T_s}\right)$$

$$T_b = \frac{T_0}{n} = \frac{8}{8} = 1\,\text{ms}$$

$$R = \frac{1}{T_b} = \frac{n}{T_0} = 1\,\text{kbits/s}$$

$$D = \frac{1}{T_s} = \frac{N}{T_0} = \frac{n}{T_0} = \frac{1}{T_b} = R = 1\,\text{kbaud}$$

$$B_{abs} = \frac{D}{2} = 500\text{ Hz} \quad \text{(Fig. 2.6)}$$

$$w(t) = \sum_{k=1}^{N} w_k \varphi_k(t) = \sum_{k=1}^{N} w_k Sa\left(\frac{\pi(t-(k-\frac{1}{2})T_s)}{T_s}\right)$$

$$w_k = w[(k-\tfrac{1}{2})T_s] = w(\tau)$$

τ : sampling time, $\tau = (k-\frac{1}{2})T_s$

Multilevel Signaling

- Example

The number of messages $M = 256$

The number of bits $n = \log_2 M = 8$ bits

The time span $T_0 = 8$ ms

$$w(t) = \sum_{k=1}^{N} w_k \varphi_k(t) \qquad 0 < t < T_0$$

N : number of symbols

For $L = 4$ multilevel signaling, $N = \frac{n}{l} = \frac{n}{\log_2 L} = \frac{8}{\log_2 4} = 4$

l : bits carried by each symbol

A particular message corresponding to the code word 01001110

$w_1 = -3,\ w_2 = -1,\ w_3 = +3,\ w_4 = +1$

Binary to Multilevel Conversion

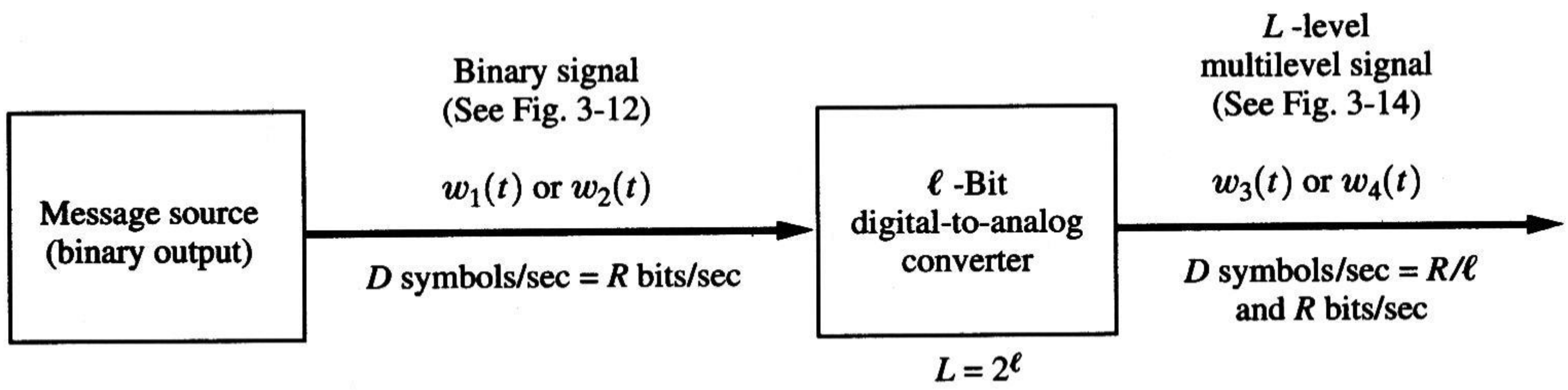
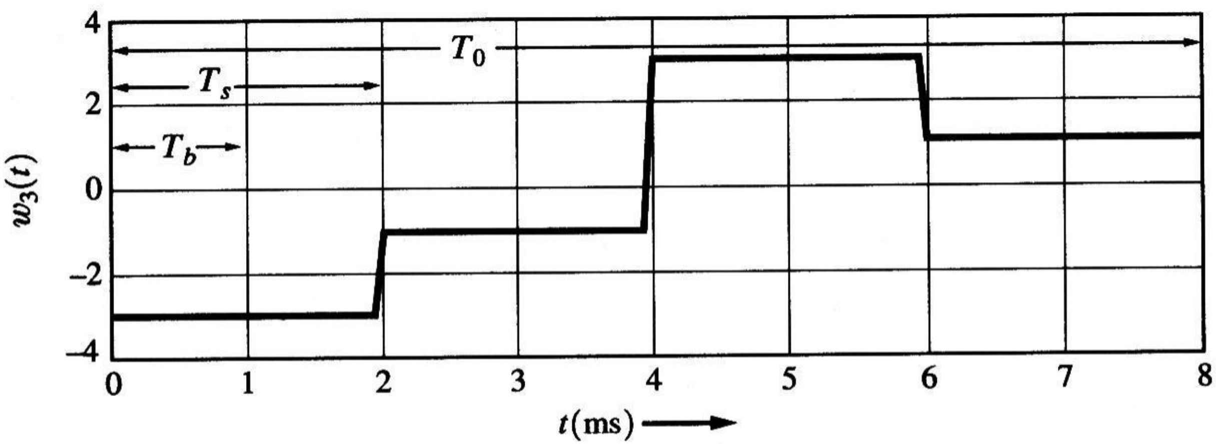


Figure 3–13 Binary-to-multilevel signal conversion.

Encoding Scheme

Binary Input (l = 2 bits)	Output Level (V)
11	+3
10	+1
00	-1
01	-3

CASE 1. RECTANGULAR PULSE ORTHOGONAL FUNCTIONS



(a) Rectangular Pulse Shape $T_b = 1$ ms

$$\varphi_k(t) = \Pi\left(\frac{t-(k-\frac{1}{2})T_s}{T_s}\right)$$

$$T_b = \frac{T_0}{n} = \frac{8}{8} = 1\,\text{ms},\;\; R = \frac{1}{T_b} = \frac{n}{T_0} = 1\,\text{kbits/s}$$

$$N = \frac{n}{l} = 4, \quad D = \frac{1}{T_s} = \frac{N}{T_0} = \frac{n/l}{T_0} = \frac{R}{l} = 500\text{ baud}$$

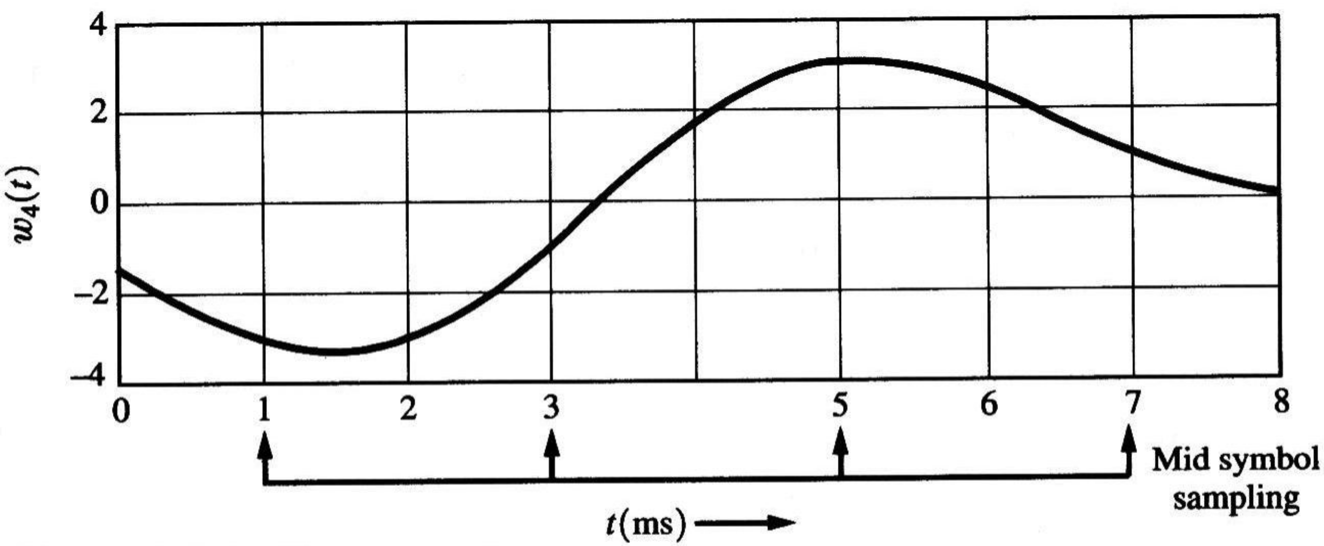
$$B_{null} = \frac{1}{T_s} = D = 500\text{ Hz}$$

$$K_k = \int_0^{T_0} \varphi_k(t)\varphi_k^*(t)dt = \int_{(k\text{-}1)T_s}^{kT_s} (1\times 1)dt = T_s$$

$$w_k = \frac{1}{K_k}\int_0^{T_0} w(t)\varphi_k^*(t)dt = \frac{1}{T_s}\int_{(k\text{-}1)T_s}^{kT_s} w(t)dt = w(\tau)$$

$$\tau:\ \text{sampling time},\ (k-1)T_s < \tau < kT_s$$

■ CASE 2. SINC PULSE ORTHOGONAL FUNCTIONS



(b) $\sin(x)/x$ Pulse Shape, $T_b = 1$ ms

Figure 3–14 $L = 4$-level signaling (computed).

$$\varphi_k(t) = \Pi\left(\frac{t-(k-\frac{1}{2})T_s}{T_s}\right)$$

$$T_b = \frac{T_0}{n} = \frac{8}{8} = 1\,\text{ms}, \quad R = \frac{1}{T_b} = \frac{n}{T_0} = 1\,\text{kbits/s}$$

$$N = \frac{n}{l} = 4, \quad D = \frac{1}{T_s} = \frac{N}{T_0} = \frac{n/l}{T_0} = \frac{R}{l} = 500\text{ baud}$$

$$B_{abs} = \frac{D}{2} = 250\text{ Hz}$$

$$w(t) = \sum_{k=1}^{N} w_k \varphi_k(t) = \sum_{k=1}^{N} w_k Sa\left(\frac{\pi(t-(k-\frac{1}{2})T_s)}{T_s}\right)$$

$$w_k = w[(k-\tfrac{1}{2})T_s] = w(\tau)$$

$$\tau:\ \text{sampling time},\ \tau = (k-\tfrac{1}{2})T_s$$

Line Codes

- Binary 1's and 0's may be represented in various serial-bit signaling formats called line codes.
- The following are some of the desirable properties of a line code:
  - Self synchronization
  - Low probability of bit error
  - A spectrum that is suitable for the channel
  - Transmission bandwidth
  - Error detection capability
  - Transparency

Binary Line Codes

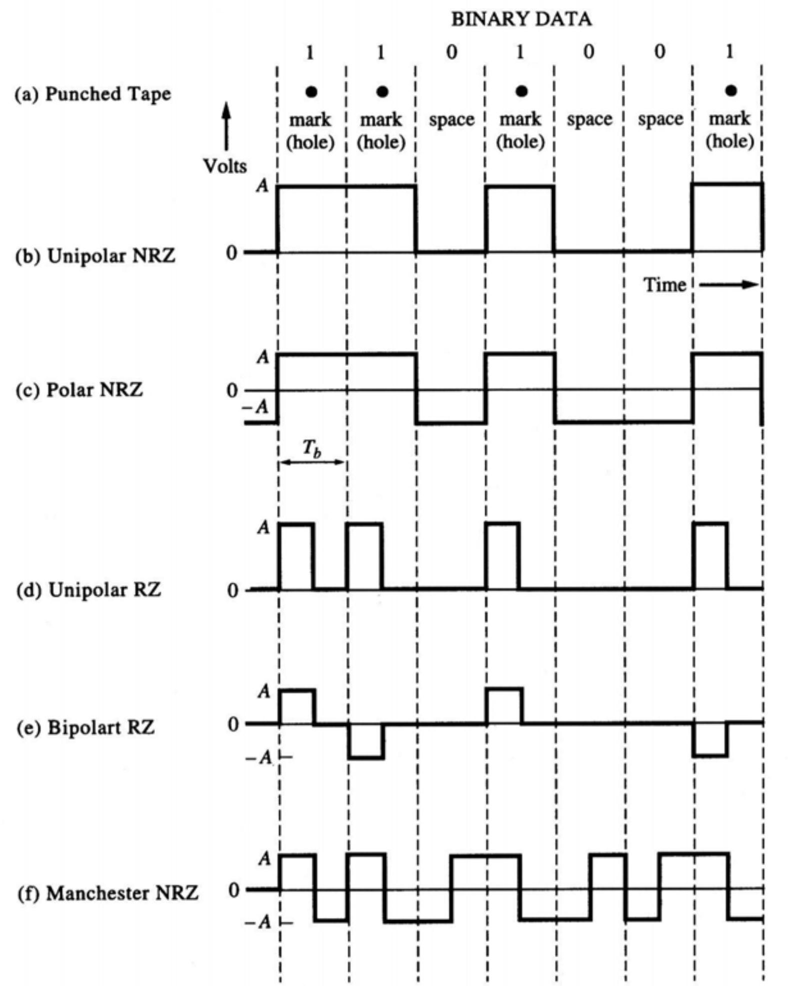


Figure 3–15 Binary signaling formats.

- A digital signal (or line code) can be represented by

$$s(t) = \sum_{n=-\infty}^{\infty} a_n f(t - nT_s)$$

$f(t)$: symbol pulse shape

T_s : symbol interval, $T_s = lT_b$

T_b : bit interval

- The general expression for the PSD of a digital signal is

$$P_s(f) = \frac{|F(f)|^2}{T_s} \sum_{k=-\infty}^{\infty} R(k) e^{j2\pi kfT_s}$$

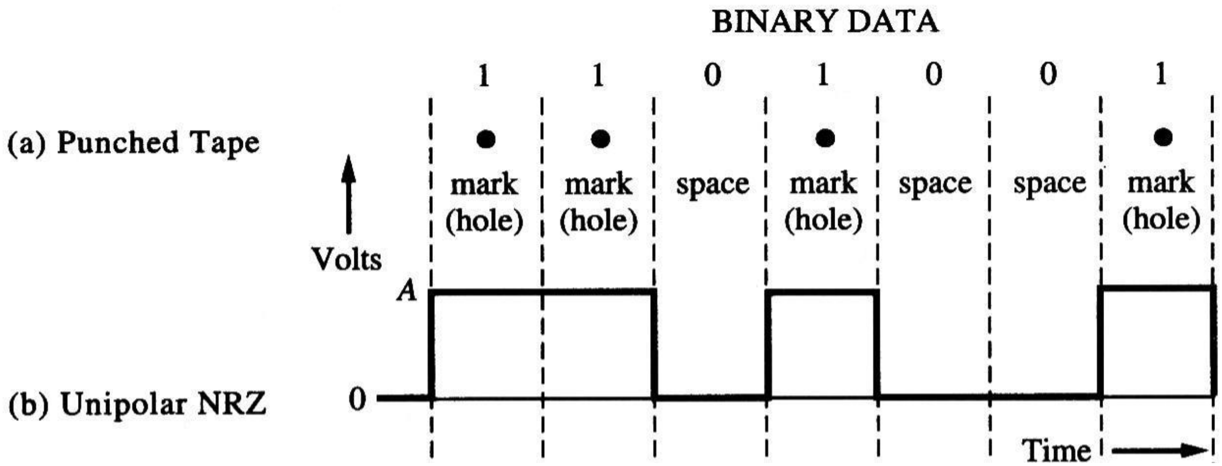
$F(f) = F[f(t)]$

$R(k)$: autocorrelation of the data

$$R(k) = \sum_{i=1}^{I} (a_n a_{n+k})_i P_i$$

P_i : probability of having the ith $a_n a_{n+k}$ product

Unipolar NRZ



$$P(a_n = A) = P(a_n = 0) = \frac{1}{2}$$

$$R(k) = \sum_{i=1}^{I} (a_n a_{n+k})_i P_i = \begin{cases} \sum_{i=1}^{2} (a_n^2)_i P_i = (A \times A) \cdot \frac{1}{2} + (0 \times 0) \cdot \frac{1}{2} = \frac{A^2}{2}, & k = 0 \\ \sum_{i=1}^{4} (a_n a_{n+k})_i P_i = (A \times A) \cdot \frac{1}{4} + (A \times 0) \cdot \frac{1}{4} + (0 \times A) \cdot \frac{1}{4} + (0 \times 0) \cdot \frac{1}{4} = \frac{A^2}{4}, & k \neq 0 \end{cases}$$

Unipolar NRZ

$$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)=\sum_{k=-\infty}^{\infty}c_n e^{j2\pi k f_0 t} = f_0\sum_{k=-\infty}^{\infty}e^{j2\pi k f_0 t} \Rightarrow$$

$$\sum_{n=-\infty}^{\infty}\delta(f-nT_0)=f_0\sum_{k=-\infty}^{\infty}e^{j2\pi k f f_0} \Rightarrow \sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T_s})=T_s\sum_{k=-\infty}^{\infty}e^{j2\pi k f T_s}$$

$$\sum_{k=-\infty}^{\infty}R(k)e^{j2\pi k f T_s}=\frac{A^2}{2}+\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{A^2}{4}e^{j2\pi k f T_s}=\frac{A^2}{4}(1+\sum_{k=-\infty}^{\infty}e^{j2\pi k f T_s})=\frac{A^2}{4}[1+\frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T_s})]$$

For rectangular pulses

$$f(t)=\Pi\left(\frac{t}{T_s}\right)\leftrightarrow F(f)=T_s Sa(\pi f T_s)$$

$$P(f)=\frac{|F(f)|^2}{T_s}\sum_{k=-\infty}^{\infty}R(k)e^{j2\pi k f T_s}=\frac{|T_s Sa(\pi f T_s)|^2}{T_s}\cdot\frac{A^2}{4}[1+\frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T_s})]$$

$$=\frac{A^2T_s}{4}Sa^2(\pi f T_s)[1+\frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T_s})]=\frac{A^2T_s}{4}Sa^2(\pi f T_s)[1+\frac{1}{T_s}\delta(f)]$$

Unipolar NRZ

$$P=\int_{-\infty}^{\infty}P(f)df=\int_{-\infty}^{\infty}\frac{A^2T_s}{4}Sa^2(\pi fT_s)[1+\frac{1}{T_s}\delta(f)]df$$

$$=\frac{A^2T_s}{4}\left(\int_{-\infty}^{\infty}Sa^2(\pi fT_s)df+\int_{-\infty}^{\infty}Sa^2(\pi fT_s)\frac{1}{T_s}\delta(f)df\right)$$

$$=\frac{A^2T_s}{4}\left(\frac{1}{\pi T_s}\int_{-\infty}^{\infty}Sa^2(x)dx+Sa^2(0)\frac{1}{T_s}\right)$$

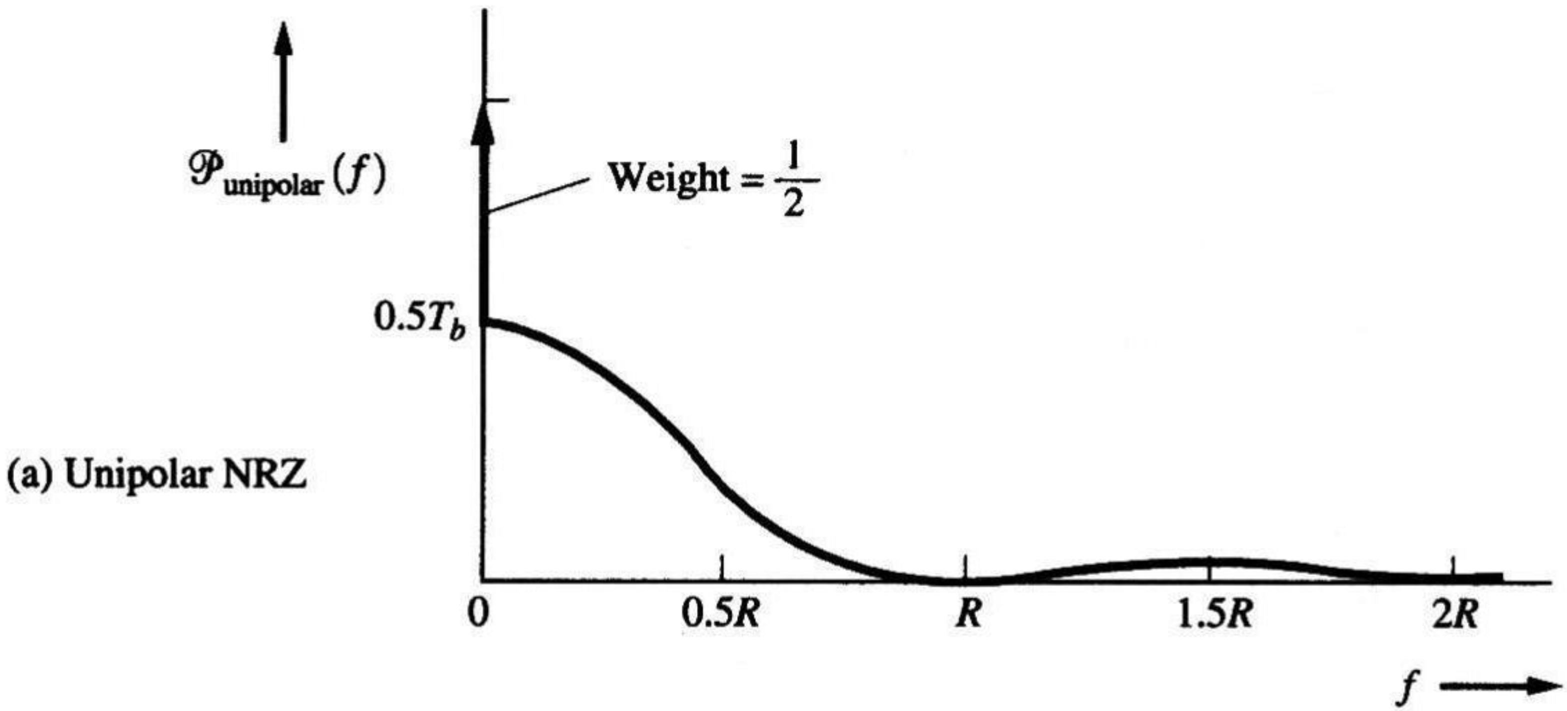
$$=\frac{A^2T_s}{4}\left(\frac{1}{\pi T_s}\cdot\pi+\frac{1}{T_s}\right)=\frac{A^2}{2}$$

$$P=1\Rightarrow A^2=2\Rightarrow A=\sqrt{2}$$

$$P(f)=\frac{2T_s}{4}Sa^2(\pi fT_s)[1+\frac{1}{T_s}\delta(f)]=\frac{T_s}{2}Sa^2(\pi fT_s)+\frac{1}{2}\delta(f)$$

$$=0.5T_bSa^2(\pi fT_b)+\frac{1}{2}\delta(f)$$

$$B_{null}=\frac{1}{T_s}=D=R$$



$\mathcal{P}_{\text{unipolar}}(f)$
Weight = $\frac{1}{2}$
$0.5T_b$
(a) Unipolar NRZ
0
0.5R
R
1.5R
2R
f

Sa Waveform NRZ

$$Sa(2\pi Wt) \leftrightarrow \frac{1}{2W}\Pi\left(\frac{f}{2W}\right)$$

For sinc pulses

$$f(t) = Sa\left(\frac{\pi t}{T_s}\right) \leftrightarrow F(f) = T_s\Pi\left(\frac{f}{1/T_s}\right)$$

$$P(f) = \frac{|F(f)|^2}{T_s}\sum_{k=-\infty}^{\infty} R(k)e^{j2\pi kfT_s} = \frac{\left|T_s\Pi\left(\frac{f}{1/T_s}\right)\right|^2}{T_s}\cdot\frac{A^2}{4}[1+\frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T_s})]$$

$$= \frac{A^2T_s}{4}\Pi\left(\frac{f}{1/T_s}\right)[1+\frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T_s})] = \frac{A^2T_s}{4}\Pi\left(\frac{f}{1/T_s}\right)[1+\frac{1}{T_s}\delta(f)]$$

$$= \frac{A^2T_s}{4}\Pi\left(\frac{f}{1/T_s}\right)[1+\frac{1}{T_s}\delta(f)]$$

$$P=\int_{-\infty}^{\infty}P(f)df=\int_{-\infty}^{\infty}\frac{A^2T_s}{4}\Pi(\frac{f}{1/T_s})[1+\frac{1}{T_s}\delta(f)]df$$

$$=\frac{A^2T_s}{4}\left(\int_{-\infty}^{\infty}\Pi(\frac{f}{1/T_s})df+\int_{-\infty}^{\infty}\Pi(\frac{f}{1/T_s})\frac{1}{T_s}\delta(f)df\right)$$

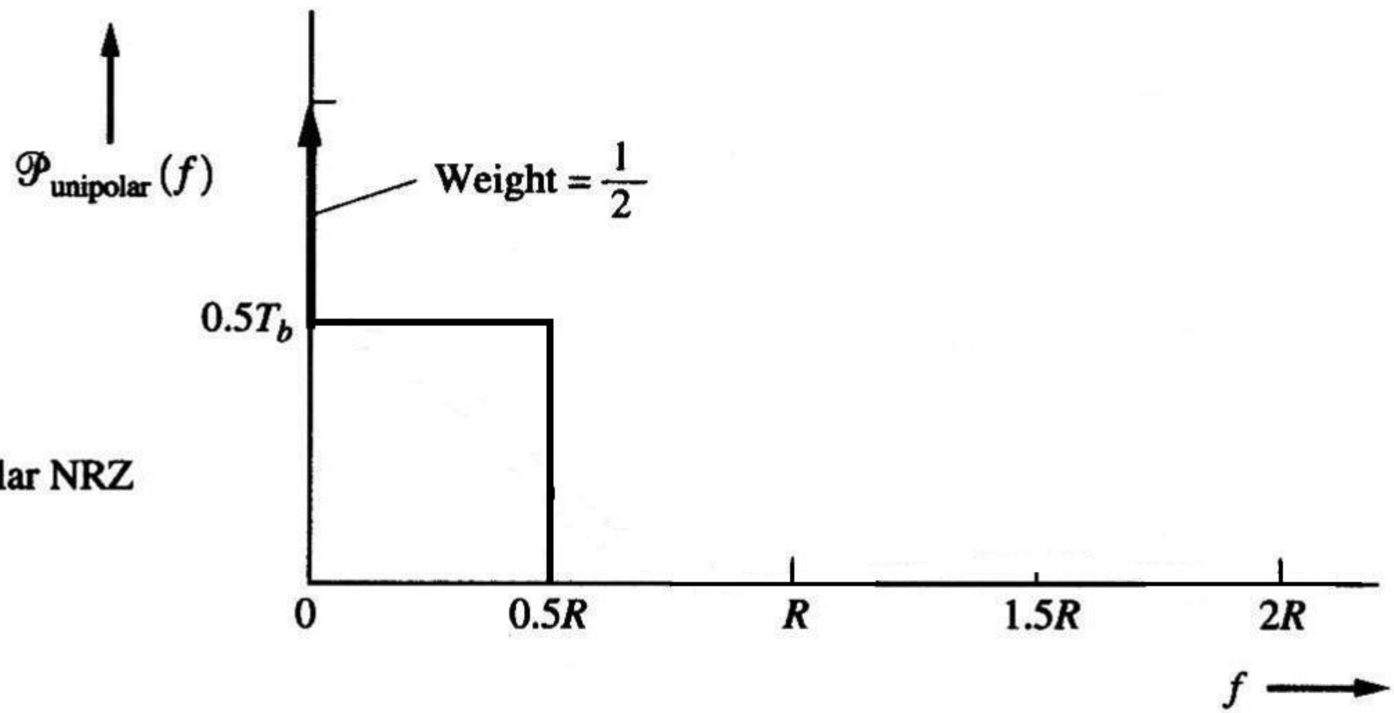
$$=\frac{A^2T_s}{4}\left(\int_{-1/2T_s}^{1/2T_s}df+\frac{1}{T_s}\right)$$

$$=\frac{A^2T_s}{4}\left(\frac{1}{T_s}+\frac{1}{T_s}\right)=\frac{A^2}{2}$$

$$P=1\Rightarrow A^2=2\Rightarrow A=\sqrt{2}$$

$$P(f)=\frac{2T_s}{4}\Pi(\frac{f}{1/T_s})[1+\frac{1}{T_s}\delta(f)]=\frac{T_s}{2}\Pi(\frac{f}{1/T_s})+\frac{1}{2}\delta(f)=0.5T_b\Pi(\frac{f}{1/T_b})+\frac{1}{2}\delta(f)$$

$$B_{abs}=\frac{1}{2T_s}=\frac{N}{2T_0}=\frac{D}{2}=\frac{R}{2}=\frac{1}{2T_b}\Rightarrow N=2B_{abs}T_0\quad\text{(dimensionality theorem)}$$

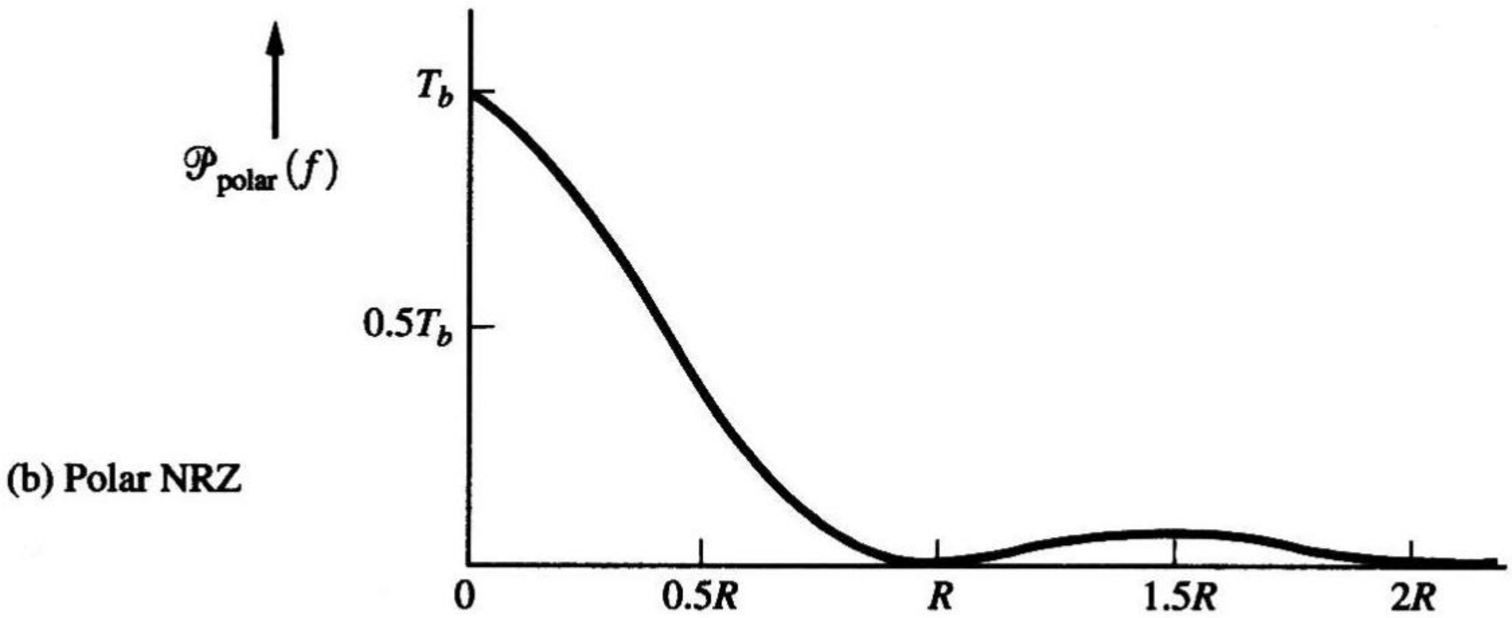


(a) Unipolar NRZ

Polar NRZ

$$P_{polar\ NRZ}(f) = A^2 T_b Sa^2(\pi f T_b)$$

$$P = 1 \Rightarrow A = 1$$

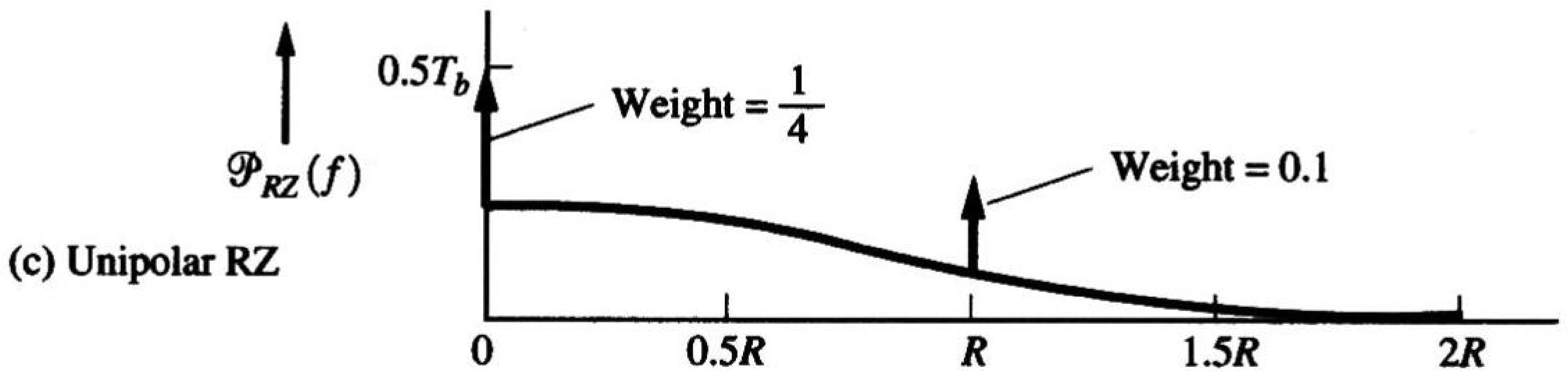


Unipolar RZ

$$d = \tfrac{1}{2}$$

$$P_{unipolar\ RZ}(f) = \frac{A^2 T_b}{16} Sa^2(\pi f T_b/2)[1 + \frac{1}{T_b}\sum_{n=-\infty}^{\infty}\delta(f - \frac{n}{T_b})]$$

$$P = 1 \Rightarrow A = 2$$

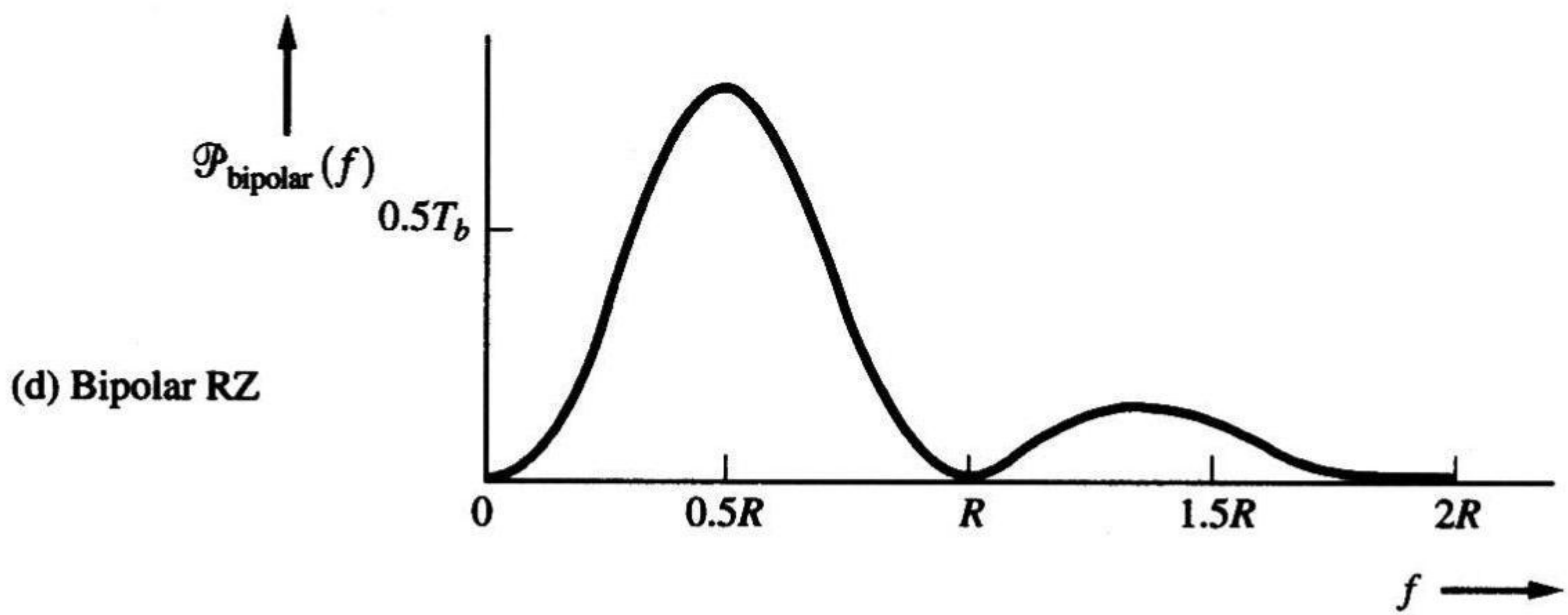


Bipolar RZ

- PSD for bipolar (also known as pseudoternary or AMI) RZ signaling

$$d = \tfrac{1}{2}$$

$$P_{bipolar\ RZ}(f) = \frac{A^2 T_b}{8} Sa^2(\pi f T_b / 2)\sin^2(\pi f T_b)$$

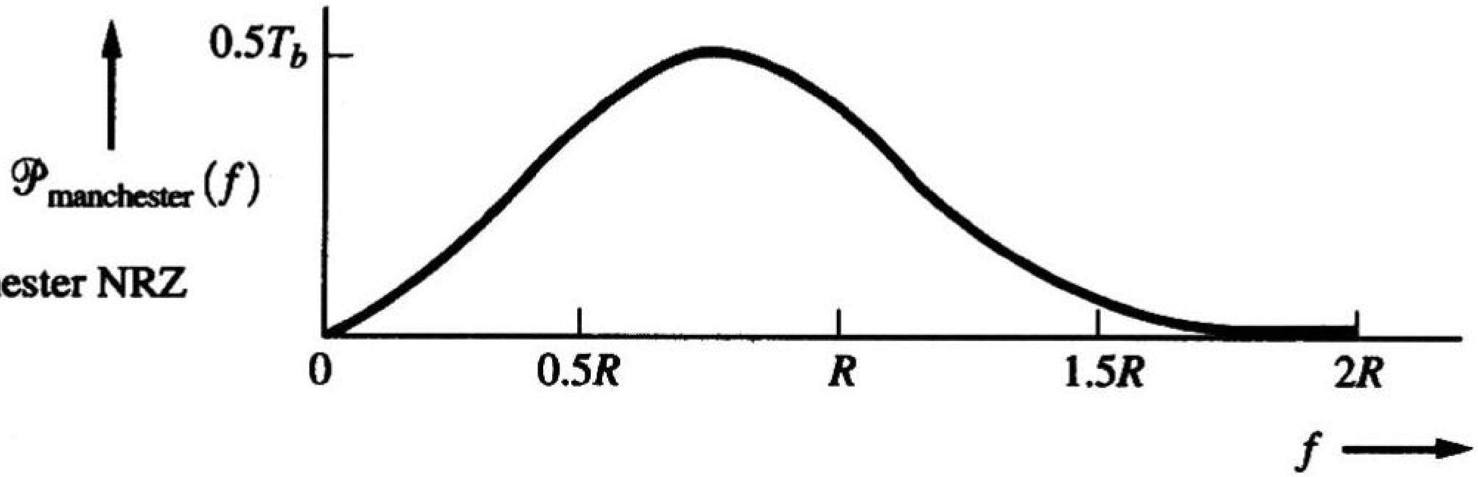


Manchester NRZ

$$f(t) = \Pi\left(\frac{t + T_b/4}{T_b/2}\right) + \Pi\left(\frac{t - T_b/4}{T_b/2}\right)$$

$$P_{Manchester\ NRZ}(f) = A^2 T_b Sa^2(\pi f T_b/2)\sin^2(\pi f T_b/2)$$

$$P = 1 \Rightarrow A = 1$$



(e) Manchester NRZ

Multilevel NRZ

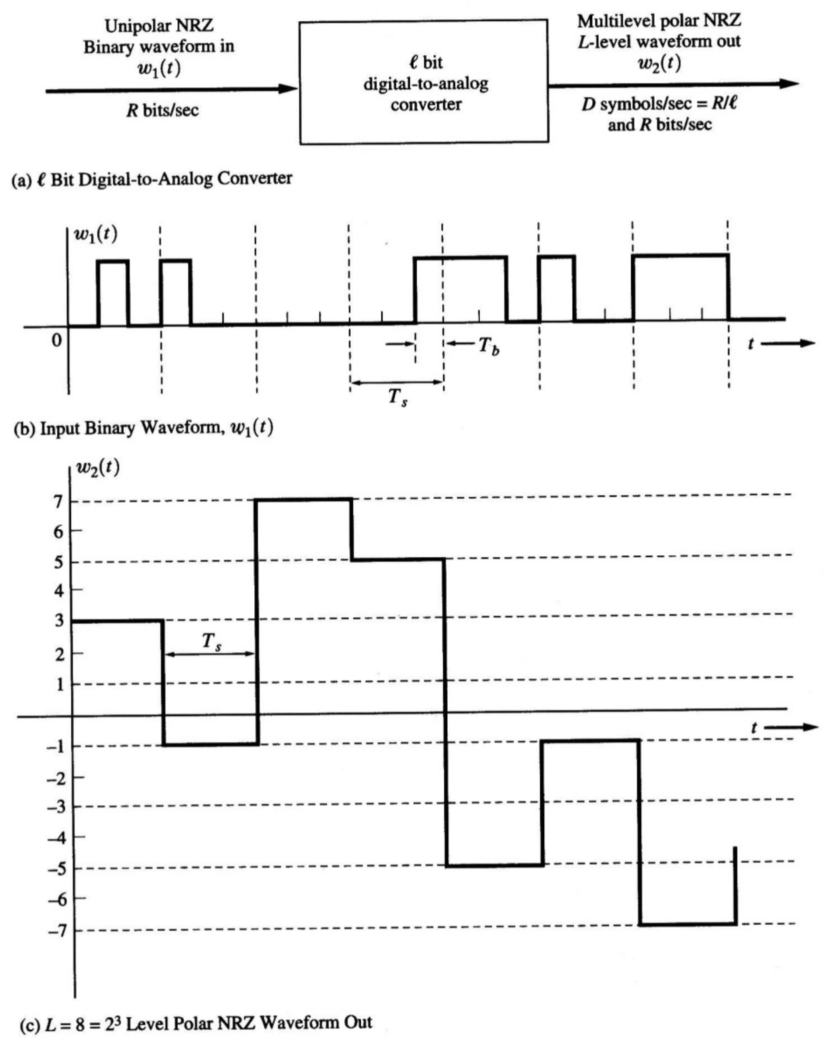


Figure 3–22 Binary-to-multilevel polar NRZ signal conversion.

$$R(0)=\sum_{i=1}^{I}(a_n)_i^2 P_i=\frac{1}{8}\sum_{i=1}^{8}(a_n)_i^2=\frac{1}{8}[7^2+5^2+3^2+1^2+(-1)^2+(-3)^2+(-5)^2+(-7)^2]=21$$

$$R(k)=\sum_{i=1}^{I}(a_n a_{n+k})_i P_i=\frac{1}{64}\sum_{i=1}^{I}(a_n a_{n+k})_i$$

$$=\frac{1}{64}[7(7+5+3+1-1-3-5-7)+5(7+5+3+1-1-3-5-7)+\cdots+(-7)(7+5+3+1-1-3-5-7)]$$

$$=0,\quad k\neq 0$$

For rectangular pulses $f(t)=\Pi\left(\frac{t}{T_s}\right)\leftrightarrow F(f)=T_s Sa(\pi f T_s)$

$$P_{8-level\ NRZ}(f)=\frac{|F(f)|^2}{T_s}\sum_{k=-\infty}^{\infty}R(k)e^{j2\pi k f T_s}=\frac{|T_s Sa(\pi f T_s)|^2}{T_s}21=21T_s Sa^2(\pi f T_s)=63T_b Sa^2(3\pi f T_b)$$

$$B_{null}=\frac{1}{T_s}=D=\frac{R}{l}$$

For sinc pulses: $P_{8-level\ NRZ}(f)=21T_s\Pi\left(\frac{f}{1/T_s}\right),\quad B_{abs}=\frac{1}{2T_s}=\frac{D}{2}=\frac{R}{2l}$ (dimensionality theorem)

- In general, for the case of a multilevel polar NRZ signal with rectangular pulse shape

$$P_{multilevel\ NRZ}(f) = KSa^2\left(l\pi f T_b\right)$$

$$B_{null} = \frac{R}{l}$$

- Multilevel signaling is used to reduce the bandwidth of a digital signal compared with the bandwidth required for binary signaling.

Spectral Efficiency

- The spectral efficiency of a digital signal is given by the number of bits per second of data that can be supported by each hertz of bandwidth. That is,

$$\eta = \frac{R}{B} \text{ (bits/s)/Hz}$$

TABLE 3–6 SPECTRAL EFFICIENCIES OF LINE CODES

Code Type	First Null Bandwidth (Hz)	Spectral Efficiency $\eta = R/B$ [(bits/s)/Hz]
Unipolar NRZ	R	1
Polar NRZ	R	1
Unipolar RZ	$2R$	$\frac{1}{2}$
Bipolar RZ	R	1
Manchester NRZ	$2R$	$\frac{1}{2}$
Multilevel polar NRZ	R/ℓ	ℓ

Homework

- LC 3-20, 3-23, 3-26, 3-32, 3-36