Baseband Digital System (1)

LC 2-7, 3-2

Lecture 12, 2008-10-21

Contents

- Sampling Theorem
- Pulse Amplitude Modulation
- Natural Sampling
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Sampling Theorem

- (Shannon) Sampling theorem. Any physical waveform can be represented over the interval $-\infty < t < \infty$ by

$$w(t) = \sum_{n=-\infty}^{\infty} a_n \frac{\sin\{\pi f_s[t-(n/f_s)]\}}{\pi f_s[t-(n/f_s)}$$

where

$$a_n = f_s \int_{-\infty}^{\infty} w(t) \frac{\sin\{\pi f_s[t-(n/f_s)]\}}{\pi f_s[t-(n/f_s)} dt$$

where f_s is a parameter that is assigned some convenient value greater than zero. Furthermore, if $w(t)$ is bandlimited to B hertz and $f_s \geq 2B$, then

$$a_n = w(n/f_s)$$

That is, for $f_s \geq 2B$ the orthogonal series coefficients are simply the values of the waveform that are obtained when the waveform is sampled every $1/f_s$ seconds.

Proof

$$\varphi_n(t) = \frac{\sin\{\pi f_s[t-(n/f_s)]\}}{\pi f_s[t-(n/f_s)}$$

$$\because \int_{-\infty}^{\infty} \varphi_n(t)\varphi_m^*(t)dt = \int_{-\infty}^{\infty} \Phi_n(f)\Phi_m^*(f)df = \int_{-\infty}^{\infty} [\frac{1}{f_s}\Pi(\frac{f}{f_s})e^{-j2\pi fn/f_s}][\frac{1}{f_s}\Pi(\frac{f}{f_s})e^{-j2\pi fm/f_s}]^* df$$

$$= \frac{1}{f_s^{\,2}} \int_{-f_s/2}^{f_s/2} e^{-j2\pi f(n-m)f/f_s} df = \begin{Bmatrix} \frac{1}{f_s}, & n=m \\ 0, & n \neq m \end{Bmatrix} = \frac{1}{f_s}\delta_{nm} = K_n\delta_{nm}$$

$$\therefore \{\varphi_n(t)\} \text{ are orthogonal with } K_n = \frac{1}{f_s}.$$

$$\therefore w(t) = \sum_n a_n\varphi_n(t) = \sum_n a_n \frac{\sin\{\pi f_s[t-(n/f_s)]\}}{\pi f_s[t-(n/f_s)}$$

$$a_n = \frac{1}{K_n}\int_{-\infty}^{\infty} w(t)\varphi_n^*(t)dt = f_s\int_{-\infty}^{\infty} w(t)\frac{\sin\{\pi f_s[t-(n/f_s)]\}}{\pi f_s[t-(n/f_s)}dt$$

Proof (con't)

If $W(f)=0, \quad \text{for } |f|>2B$ then

$$a_n = \frac{1}{K_n}\int_{-\infty}^{\infty} w(t)\varphi_n^*(t)dt = \frac{1}{K_n}\int_{-\infty}^{\infty} W(f)\Phi_n^*(f)df$$

$$= f_s \int_{-\infty}^{\infty} W(f)[\frac{1}{f_s}\Pi(\frac{f}{f_s})e^{-j2\pi fn/f_s}]^* df = \int_{-\min(B,f_s/2)}^{\min(B,f_s/2)} W(f)e^{j2\pi fn/f_s}df$$

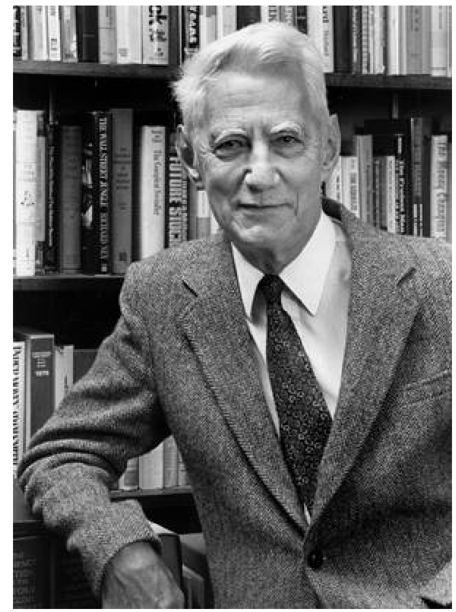
$$= \int_{-B}^{B} W(f)e^{j2\pi fn/f_s}df = \int_{-\infty}^{\infty} W(f)e^{j2\pi fn/f_s}df = w(n/f_s)$$

- It is obvious that the sampling rate allowed to reconstruct a bandlimited waveform without error is given by, Nyquist Frequency

$$f_s \geq 2B$$

People

Harry Nyquist
Nyquist Theorem (1928)



Claude Elwood Shannon
Information Theory (1948)

Waveform Reconstructed from Sampling Values

- Suppose that we are interested in reproducing the waveform over a T_0 interval. Then we can truncate the sampling function series so that we include only N of the $\varphi_n(t)$ functions that have their peaks within the T_0 interval of interest.

$$w(t) = \sum_{n=-\infty}^{\infty} w(n/f_s) \frac{\sin\{\pi f_s[t-(n/f_s)]\}}{\pi f_s[t-(n/f_s)}$$

$$\approx \sum_{n=n_1}^{n_1+N-1} w(n/f_s) \frac{\sin\{\pi f_s[t-(n/f_s)]\}}{\pi f_s[t-(n/f_s)}$$

The minimum number of samples $w(n/f_n)$ that are needed to reconstructed the waveform is

$$N = \frac{T_0}{1/f_s} = f_s T_0 \geq 2BT_0$$

Example (1)

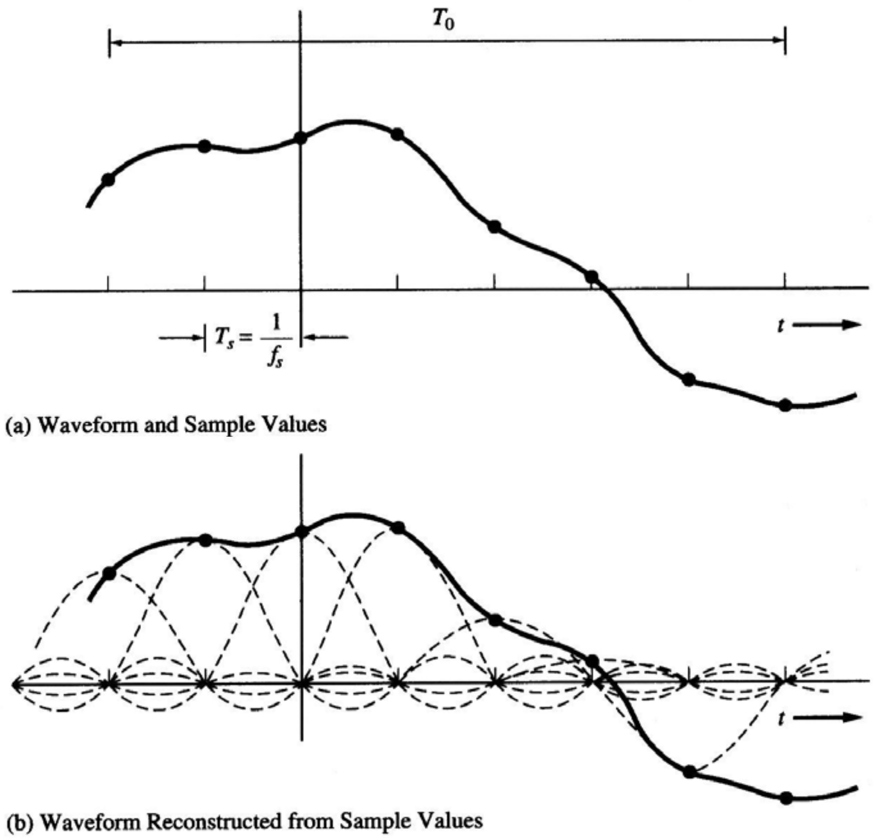
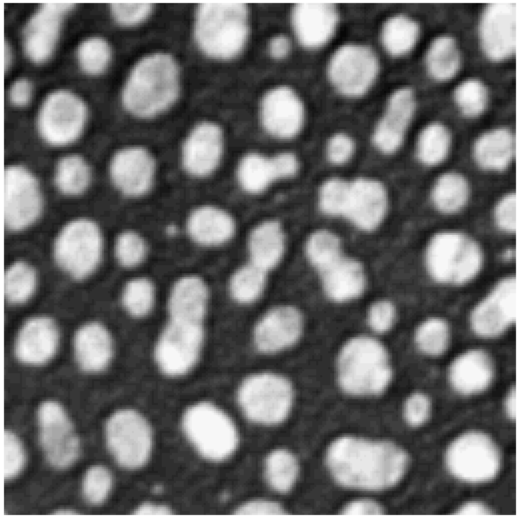


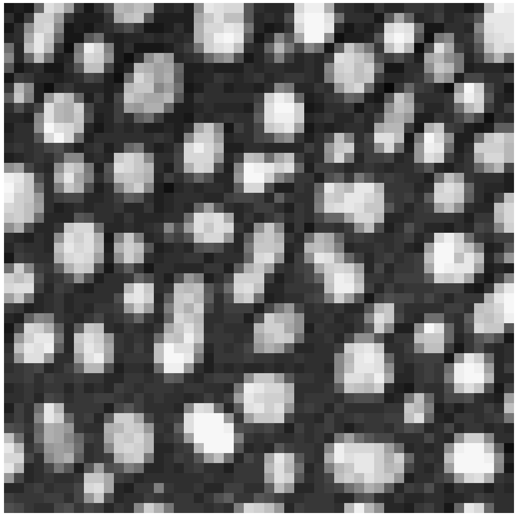
Figure 2–17 Sampling theorem.

Example (2)

Nyquist sampling in Digital Microscopy



Appropriate Sampling



Under-Sampling

Pulse Amplitude Modulation

- Pulse amplitude modulation (PAM) is an engineering term that is used to describe the conversion of the analog signal to a pulse-type signal in which the amplitude of the pulse denotes the analog information.
- If $w(t)$ is an analog waveform bandlimited to B, the PAM signal that uses natural sampling (gating) is

$$w_s(t) = w(t)s(t) = w(t)\sum_{k=-\infty}^{\infty} \Pi\left(\frac{t-kT_s}{\tau}\right)$$

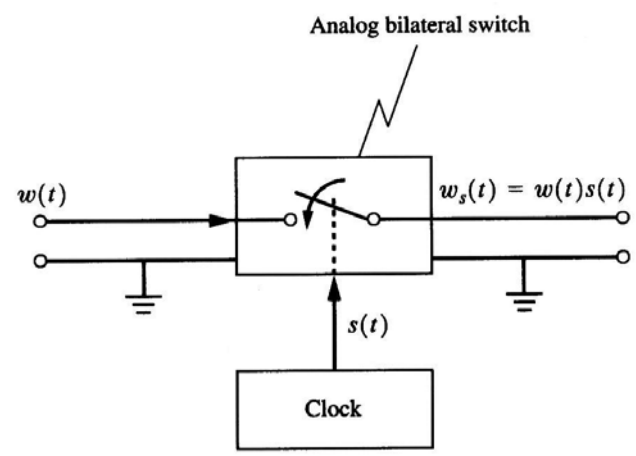


Figure 3–2 Generation of PAM with natural sampling (gating).

Example

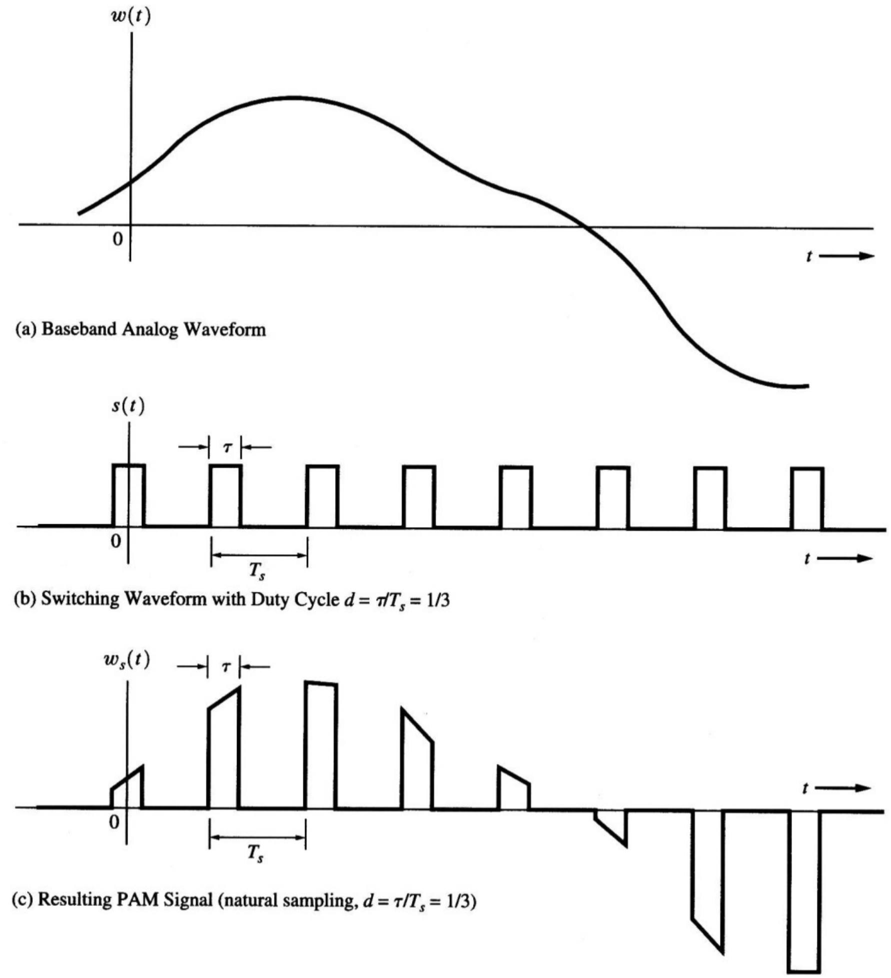
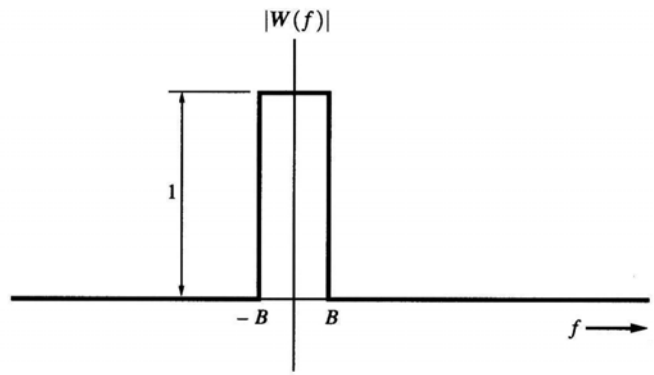


Figure 3–1 PAM signal with natural sampling.

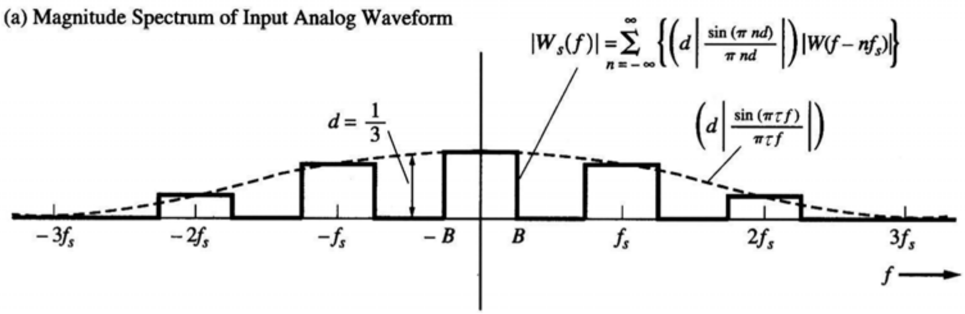
Spectrum of Naturally Sampled PAM

- The spectrum for a naturally sampled PAM signal is

$$W_s(f) = F[w_s(t)] = d \sum_{n=-\infty}^{\infty} Sa(\pi n d) W(f - n f_s)$$



(a) Magnitude Spectrum of Input Analog Waveform



(b) Magnitude Spectrum of PAM (natural sampling) with $d = 1/3$ and $f_s = 4$ B

Figure 3–3 Spectrum of a PAM waveform with natural sampling.

Proof

$$w_s(t) = w(t)s(t)$$

$$W_s(f) = W(f) * S(f) = W(f) * \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_s) \qquad \text{(2-109)}$$

$$= \sum_{n=-\infty}^{\infty} c_n W(f) * \delta(f - nf_s)$$

$$= \sum_{n=-\infty}^{\infty} c_n W(f - nf_s)$$

$$c_n = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \Pi\left(\frac{t}{\tau}\right) e^{-jn\omega_0 t} dt = \frac{1}{T_s}\tau Sa(\pi \frac{n}{T_s}\tau) = dSa(\pi nd) \qquad \text{(2-89)}$$

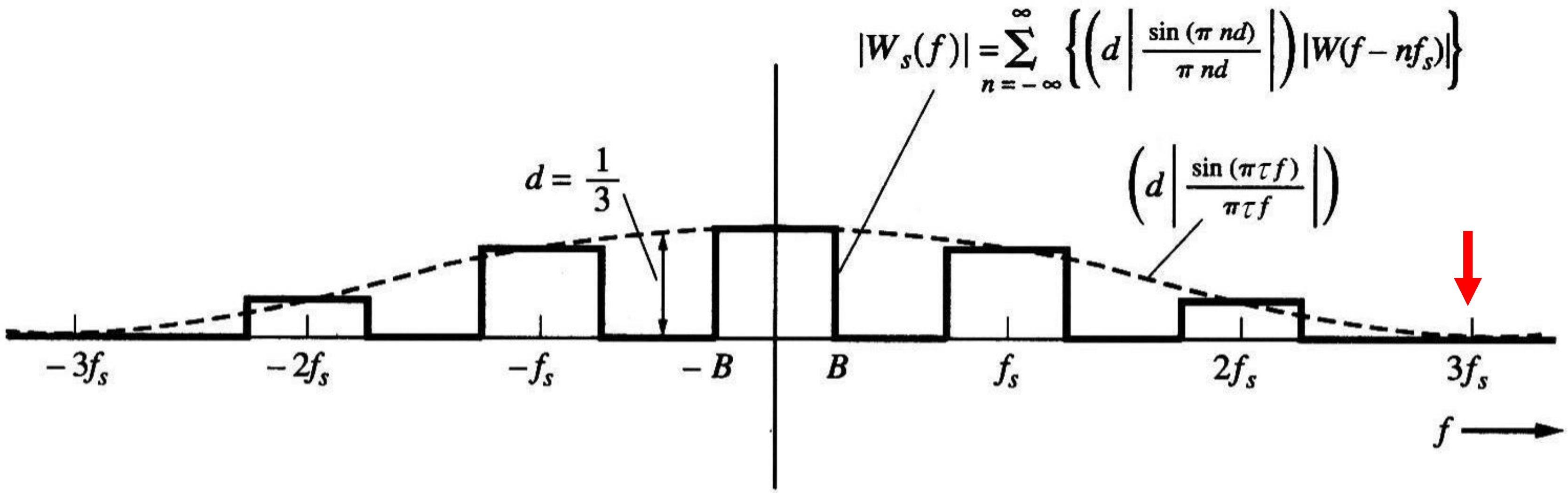
$$W_s(f) = d\sum_{n=-\infty}^{\infty} Sa(\pi nd) W(f - nf_s)$$

Example

- The bandwidth of the PAM signal is much larger than the bandwidth of the original analog signal

$$B_{null} = \frac{1}{\tau} = \frac{1}{dT_s} = \frac{f_s}{d}$$

$$B_{null} = \frac{4B}{1/3} = 12B \quad (\text{In this example}, d = 1/3,\ f_s = 4B)$$

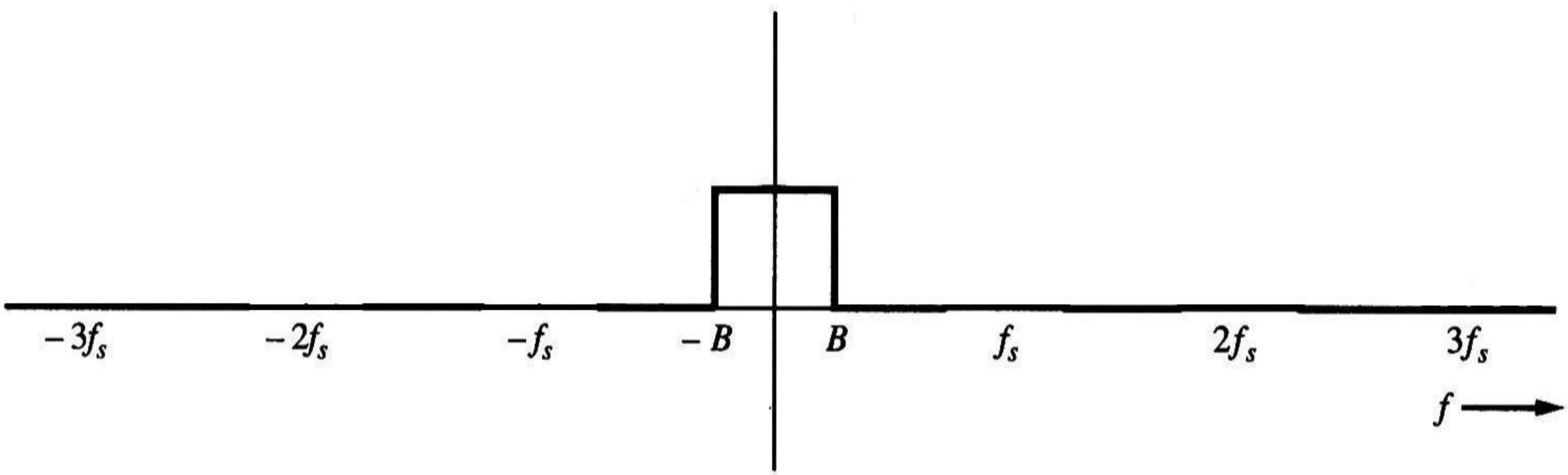


(b) Magnitude Spectrum of PAM (natural sampling) with $d = 1/3$ and $f_s = 4$ B

PAM Demodulation

- At the receiver, *w*(*t*) can be recovered by passing the PAM signal through an LPF

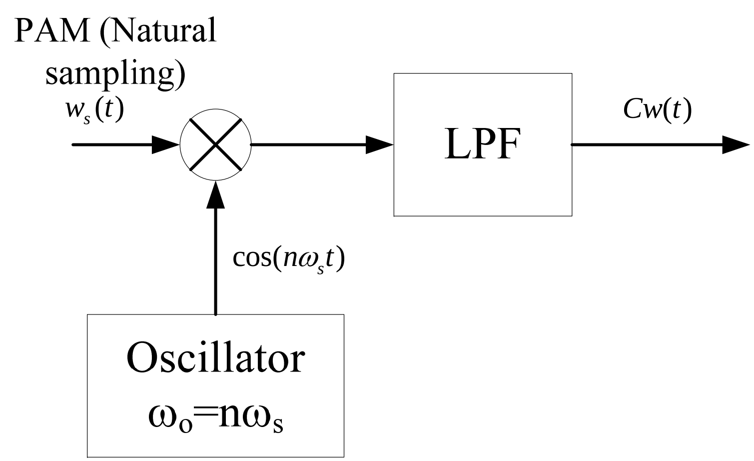
$$W(f) = W_s(f)\Pi(\frac{f}{f_{cutoff}}) \quad (B < f_{cutoff} < f_s - B)$$



(b) Magnitude Spectrum of PAM (natural sampling) with $d = 1/3$ and $f_s = 4$ B

PAM Demodulation

- Why we need a product detector when a simple low-pass filter can work?
- Because of noise on the PAM signal. Noise due to power supply hum or mechanical circuit vibration might fall in the band corresponding to n=0, and Other bands might be relatively noise free.



Instantaneous Sampling

- If $w(t)$ is an analog waveform bandlimited to B, the PAM signal that uses instantaneous sampling is

$$w_s(t) = \sum_{k=-\infty}^{\infty} w(kT_s)h(t - kT_s)$$

where h(t) denotes the sampling-pulse.

$$w_s(t) = \sum_{k=-\infty}^{\infty} w(kT_s)\Pi(\frac{t - kT_s}{T_s})$$

Example

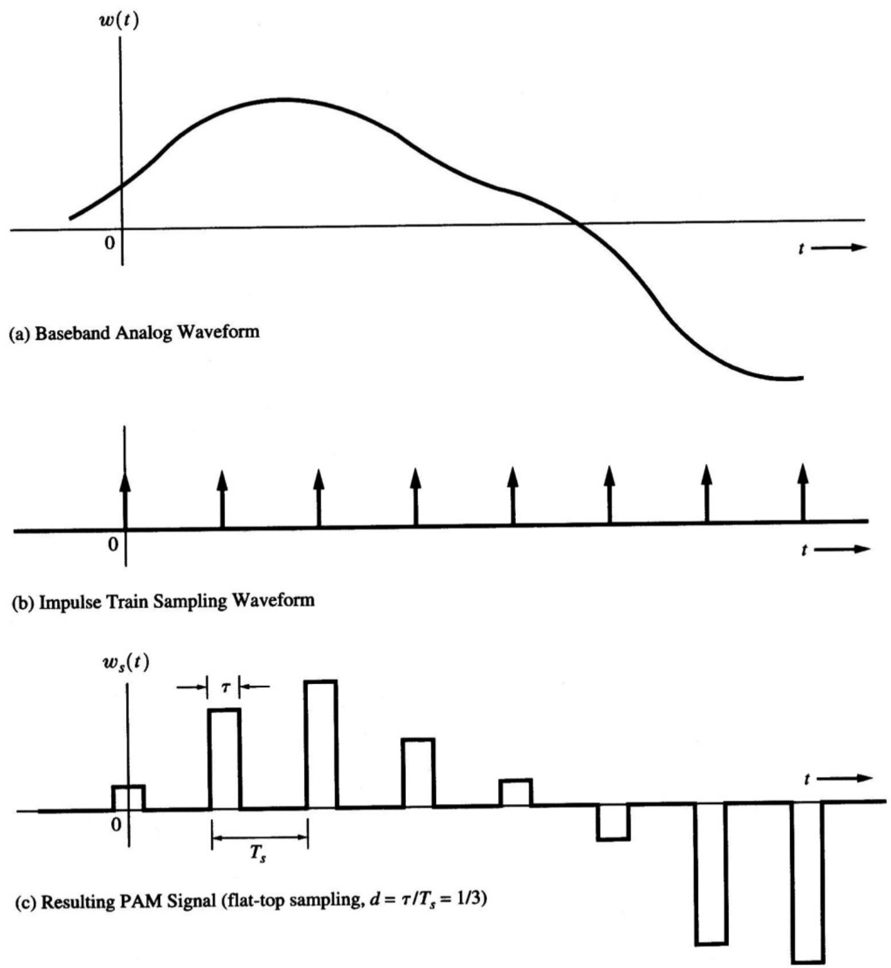


Figure 3–5 PAM signal with flat-top sampling.

Spectrum of Instantaneous Sampling PAM

- The spectrum for a instantaneous sampled PAM signal is

$$W_s(f) = F[w_s(t)] = \frac{1}{T_s} H(f) \sum_{n=-\infty}^{\infty} W(f - nf_s) = \frac{\tau}{T_s} Sa(\pi f \tau) \sum_{n=-\infty}^{\infty} W(f - nf_s)$$

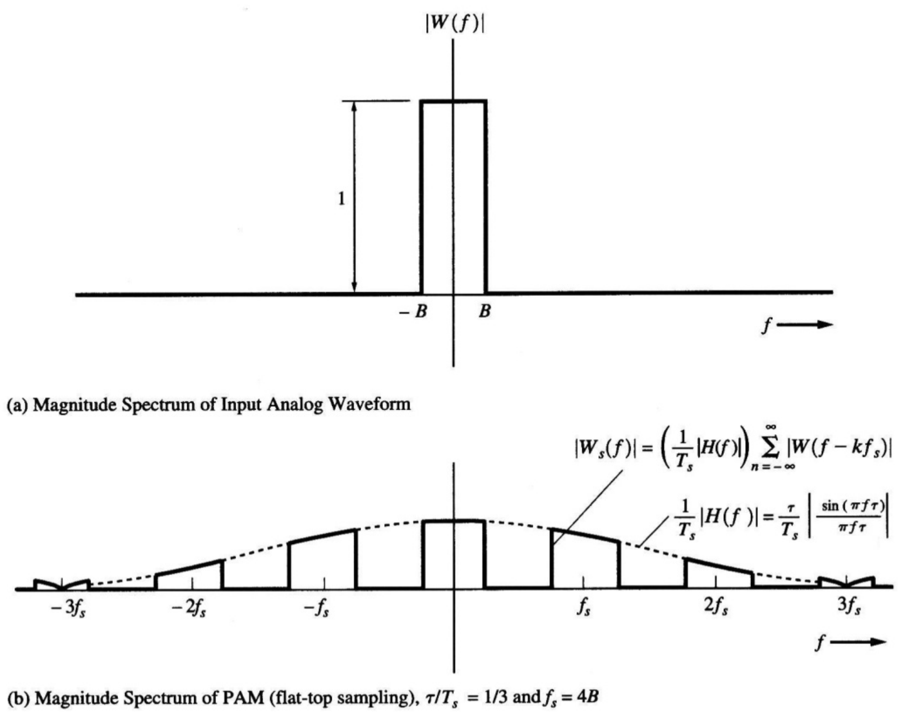


Figure 3–6 Spectrum of a PAM waveform with flat-top sampling.

Proof

$$w_s(t) = \sum_k w(kT_s)h(t - kT_s) = \sum_k w(kT_s)[h(t) * \delta(t - kT_s)]$$

$$= \sum_k h(t) * [w(kT_s)\delta(t - kT_s)] = \sum_k h(t) * [w(t)\delta(t - kT_s)]$$

$$= h(t) * [\sum_k w(t)\delta(t - kT_s)]$$

$$= h(t) * [w(t)\sum_k \delta(t - kT_s)]$$

$$W_s(f) = H(f)[W(f) * f_s \sum_n \delta(f - kf_s)]$$

$$= H(f) f_s [W(f) * \sum_n \delta(f - kf_s)]$$

$$= H(f) f_s [\sum_n W(f) * \delta(f - kf_s)] = H(f) f_s \sum_n W(f - kf_s)$$

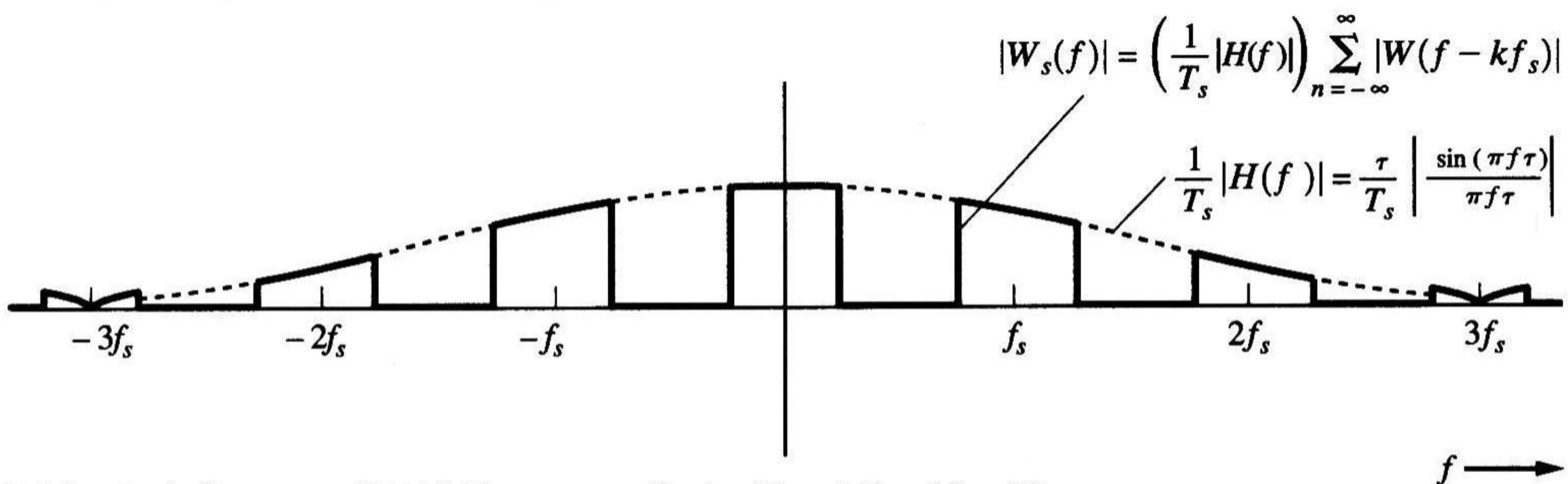
$$= \frac{1}{T_s} H(f) \sum_n W(f - kf_s)$$

Example

- The bandwidth of the PAM signal is much larger than the bandwidth of the original analog signal

$$B_{null} = \frac{1}{\tau}$$

$$B_{null} = \frac{1}{\frac{1}{3}T_s} = 3f_s = 12B \quad (\text{In this example}, d = 1/3,\ f_s = 4B)$$



(b) Magnitude Spectrum of PAM (flat-top sampling), $\tau/T_s = 1/3$ and $f_s = 4B$

Recover from PAM

- At the receiver, $w(t)$ can be recovered by passing the PAM signal through an LPF

$$W(f) = W_s(f)\Pi(\frac{f}{f_{cutoff}}) \quad (B < f_{cutoff} < f_s - B)$$

- There is some high-frequency loss in the recovered analog waveform. This loss, if significant, can be reduced by decreasing τ, or by using some additional gain at the high frequencies in the LPF transfer function, the LPF would be called an equalization filter and have a transfer function of

$$\frac{1}{H(f)}$$

- The pulse width τ is also called the aperture since τ / T_s determines the gain of the recovered analog signal, which is small if τ is small relative to T_s.

Summary

- The transmission of either naturally or instantaneously sampled PAM over a channel requires a very wide frequency response because of the narrow pulse width, which imposes stringent requirements on the magnitude and phase response of the channel.
- The bandwidth required is much larger than that of the original analog signal, and the noise performance of the PAM system can never be better than the analog system. Therefore, PAM is not very good for long-distance tranmission.
- It provides a method for A/D conversion.
- Multiple PAM signals from different sources can be interleaved to transmit all of the information over a single channel. TDM

Homework

- LC 3-2, 3-3, 3-4, 3-5